15.5 – Probability
University of Massachusetts Amherst
Math 233 – Fall 2013

Definition 1. A function $f(x,y)$ is a *continuous probability distribution function* if

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$

Probability distribution functions give us a means for computing the probability of a given event. Namely, if x and y have the joint distribution $f(x,y)$, then the probability that $a \le x \le b$ and $c \le y \le d$ is defined to be

$$P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Example 1.

(i) Find C so that $f(x,y)$ is a probability distribution function, where $f(x,y)$ is the function

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \le x \le 10 \text{ and } 0 \le y \le 10. \\ 0 & \text{otherwise.} \end{cases}$$

(ii) A marble is thrown into a sandbox measuring 10×10 meters. Take (0,0), (10,0), (10,10), and (0,10) to be the corners of the sandbox. The final position of the marble is given by the distribution $f(x,y)$ from part (i). What is the probability that the marble is in the lower left corner of the box? (That is, $0 \le x \le 5$ and $0 \le y \le 5$.)

Definition 2. If x and y have the joint distribution function $f(x,y)$, the *expected value* (or *mean*, or *average value*) of x and y are

$$E(x) = \iint\limits_{\mathbb{R}^2} x f(x,y)\, dA, \qquad\qquad E(y) = \iint\limits_{\mathbb{R}^2} y f(x,y)\, dA.$$

Example 2. What is the "most likely" location of the marble in the previous example?

Example 3. The *bell curve* is modeled by the *Gaussian distribution*

$$\int_{-\infty}^{\infty} Ce^{-x^2}\,dx.$$

Since we cannot write an antiderivative for e^{-x^2}, we cannot evaluate this integral directly to compute C. Instead, we determine C by considering the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy. \tag{1}$$

(i) Show that the double integral in equation (1) is a square.

(ii) Rewrite the double integral in equation (1) using polar coordinates and evaluate this integral.

(iii) What is C?