# 16.3 – The Fundamental Theorem for Line Integrals
University of Massachusetts Amherst
Math 233 – Fall 2013

Recall from single variable calculus the Fundamental Theorem of Calculus: if $F'(x)$ is continuous on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

A similar statement is true for line integrals.

**Theorem 1.** *Let $C$ be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let $f$ be a differentiable function of two or three variables whose gradient vector $\nabla f$ is continuous on $C$. Then*

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

*Proof.* For functions of three variables,

$$\begin{aligned}
\int_C \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \vec{r}\,'(t)\,dt &= \int_C \left( \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \\
&= f(\vec{r}(b)) - f(\vec{r}(a))
\end{aligned}$$

□

**Note:** The above theorem is also true for piecewise-smooth curves.

In physical problems, such as computing the work done by a force $\vec{F}$ in the previous section, it is sometimes the case that there is a function $f$ such that $\vec{F} = \nabla f$.

**Example 1.** Find the work done by the gravitational field

$$\vec{F}(x, y, z) = -\frac{mMG}{|\langle x, y, z \rangle|^3} \langle x, y, z \rangle$$

in moving a particle with mass $m$ from the point (2,2,1) to the point (3,0,4) along a piecewise-smooth curve $C$.

**Corollary 2.** *If $C_1$ and $C_2$ are piecewise-smooth curves (or* paths*) with the same initial point $A$ and terminal point $B$, then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ whenever $\nabla f$ is continuous.*

**Definition 1.** If $\vec{F}$ is a continuous vector field with domain $D$, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is *independent of path* if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths $C_1$ and $C_2$ in $D$ that have the same initial and terminal points.

**Definition 2.** A vector field $\vec{F}$ is called a *conservative vector field* if it is the gradient of some scalar function, that is, there exists a function $f$ such that $\vec{F} = \nabla f$. The function $f$ is then referred to as a *potential function* for $\vec{F}$.

**Note:** It follows from the definitions and the corollary that line integrals of conservative vector fields are independent of path.

**Theorem 3.** *We say a path is* closed *if its terminal point coincides with its initial point. $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$ if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path $C$ in $D$.*

**Picture proof / open set and connected set definition.**

**Theorem 4.** *Suppose $\vec{F}$ is a vector field that is continuous on an open connected region $D$. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in $D$, then $\vec{F}$ is a conservative vector field on $D$; that is, there exists a function $F$ such that $\nabla f = \vec{F}$.*

**Theorem 5.** *If $\vec{F}(x,y) = \langle\, P(x,y), Q(x,y)\,\rangle$ is a conservative vector field, where $P$ and $Q$ have continuous first-order partial derivatives on a domain $D$, then throughout $D$ we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

**Why?**

**Picture definitions: simple curve, simply-connected region.**

**Theorem 6.** *Let $\vec{F}(x,y) = \langle\, P(x,y), Q(x,y)\,\rangle$ be a vector field on an open simply-connected region $D$. Suppose that $P$ and $Q$ have continuous first order derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

*throughout $D$. Then $\vec{F}$ is conservative.*

**Example 2.** Is $\vec{F}(x,y) = \left\langle \ln y + 2xy^3, 3x^2y^2 + x/y \right\rangle$ a conservative vector field? If so, find a function $f$ such that $\vec{F} = \nabla f$.

**Example 3.** Find the work done by the force field $\vec{F}(x,y) = \left\langle 2y^{3/2}, 3x\sqrt{y} \right\rangle$ in moving an object from (1,1) to (2,4).

**Example 4.** Is $D = \{(x,y) \mid 1 < |x| < 2\}$ open, connected, simply-connected, or none of the above?