Sections 2.1–2.4
Logical statements I

Definition

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For a statement be true, it must be true unconditionally. For example, the following statements are irrefutably true.

- $2 \in \mathbb{Z}$.
- $\mathbb{N} \subseteq \mathbb{Z}$.
- $\{0, 1, 2\}$ is a set of three elements.
- The first month of the year is January.
- We live on Earth.

Definition

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Statements that are sometimes true, or never true, are false. For example, these statements are false.

- $\mathbb{N} \in \mathbb{Z}$ ($\mathbb{N}$ is not an integer)
- $1/3 = 0.33333$ ($\frac{1}{3} \neq \frac{33333}{100000}$)
- If $a > b$, then $a^2 > b^2$. (False if $a < 0$.)
- Winter days in New England are bitterly cold. (Some days.)
- You know nothing, Jon Snow. (He knows winter is coming.)

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We may not always have enough information to decide if a statement is true or false.

- There is intelligent life elsewhere in the universe.
- Every even integer, except 2, is the sum of two primes.

$$\begin{aligned} 4 &= 2+2 & \qquad 124 &= 113+11 \\ 6 &= 3+3 & 126 &= 113+13 \\ 8 &= 3+5 & 128 &= 109+19 \\ 10 &= 5+5 & 130 &= 127+3 \\ &\vdots & &\vdots \end{aligned}$$

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Not every sentence is a statement. For example,

- Let A be a set.
- Consider the subsets of $\mathbb{Q}$.
- Suppose it is raining.
- Did you get eggs at the store?

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- $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{R}$.

  This statement is true because

  - $\mathbb{N} \subseteq \mathbb{Z}$ is true, and
  - $\mathbb{Z} \subseteq \mathbb{R}$ is true.

- $0 \in \mathbb{Z}$ and $0 \in \mathbb{N}$.

  This statement is false because

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P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

- $\wedge$ is the 'logical and' symbol. It looks like an 'A' for 'and'.

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  This statement is true because
  - $0 \in \mathbb{Z}$ is true.

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T	T	T
T	F	T
F	T	T
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- $\vee$ is the 'logical or' symbol.

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Examples:

- P: I baked a pie.
  $\sim P$: I did not bake a pie.
- P: $3 \notin \mathbb{Q}$.
  $\sim P$: $3 \in \mathbb{Q}$.
- P: Today is a week day.
  $\sim P$: It is the weekend.
- P: $x < 4$.
  $\sim P$: $x \geq 4$.

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P	$\sim P$ (or $\neg P$)
T	F
F	T

- $\sim$ (or $\neg$) is the negation symbol.

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P	Q	$P \Rightarrow Q$
T	T	T

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Case 1. You pass the exam and the course.

Case 2. You pass the exam and fail the course.

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Case 2. You pass the exam and fail the course.

Case 3. You fail the exam and pass the course.

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Example: If you pass the final exam, then you will pass the course.

Case 1. You pass the exam and the course.

Case 2. You pass the exam and fail the course.

Case 3. You fail the exam and pass the course.

Case 4. You fail the exam and fail the course.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\Rightarrow$ is the symbol for 'implies'.

"If P, then Q" statements can take on a variety of forms.

Phrase	Example
If P, then Q.	If it rains heavily, then the rivers will swell.
Q whenever P.	The rivers swell whenever it rains heavily.
P implies Q.	That it is raining heavily implies that the rivers will swell. (It rained heavily, so the rivers will be swollen.)
Q is implied by P.	That the rivers are swollen is implied by the fact that it rained heavily. (The rivers are swollen because it rained heavily.)
P is sufficient for Q.	Heavy rain is a sufficient reason for the rivers to swell.
Q is necessary for P.	Swollen rivers are a necessary consequence of heavy rain.
P only if Q.	It rained heavily only if the rivers swelled.
Q if P.	The rivers swell if it rains heavily.

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- If P, then Q: If we are in class, then it is a school day. (True.) If Q, then P: If it is a school day, then we have class. (False, we do not have class on Thursdays.)

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  If Q, then P: If $x = 1$, then $x^2 = 1$. (True.)

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  If Q, then P: If $x = 1$, then $x^2 = 1$. (True.)
- If P, then Q: If today is Tuesday, then tomorrow is Wednesday. (True.)
  If Q, then P: If tomorrow is Wednesday, then today is Tuesday. (True.)

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If Q, then P: If tomorrow is Wednesday, then today is Tuesday.

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If Q, then P: If tomorrow is Wednesday, then today is Tuesday.

P if and only if Q: Today is Tuesday if and only if tomorrow is Wednesday.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$\Leftrightarrow$ is the symbol for 'if and only if'.

Homework.

1. Read Section 2.1–2.4
2. Write up the following exercises.
   Section 2.1: 6–10. For each false statement, give a one sentence explanation as to why the statement is false.
   Section 2.2: 7–12.
   Section 2.3: 10–13.
   Section 2.4: 1, 2.

New LaTeX commands

$\land$	`\land`
$\lor$	`\lor`
$\sim$	`\sim`
$\neg$	`\neg`
$\Rightarrow$	`\implies` (`\Rightarrow`)
$\Leftrightarrow$	`\iff` (`\Leftrightarrow`)