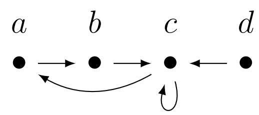
MATH 2001
INVERSES

Definition. The set R is a relation from the set A to the set B if ...

Definition. Let R be a relation from A to B. The *inverse relation* (denoted R^{-1}) is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

⋆ **Exercise 1.** Let $A = \{a, b, c, d\}$ and consider the relation R from A to itself given by the following diagram. Write the sets R and R^{-1} (using the proper notation).



Definition. Suppose A and B are sets, and f is a relation from A to B. The relation f is a *function* if ...

⋆ **Exercise 2.** Let $A = \{n \in \mathbb{Z} : |n| \leq 2\}$, and let $f : A \to \mathbb{R}$ be defined by $f(x) = x^2$. Write the sets f and f^{-1}. Is f^{-1} a function? Explain why or why not.

Definition. Let $f : A \to B$ be a relation and suppose $U \subseteq A$. Then the *image* of U in B is the set

$$f(U) = \{f(x) \in B : x \in U\}.$$

Definition. Let $f : A \to B$ be a function and suppose $V \subseteq B$. Then the *inverse image* of V in A is the set

$$f^{-1}(V) = \{x \in A : f(x) \in V\}.$$

⋆ **Exercise 3.** Let f and A be defined as in Exercise 2. Write out each of the following sets:

a.) $f(A)$

b.) $f(\mathbb{N} \cap A)$

c.) $f^{-1}(\mathbb{R})$

d.) $f^{-1}(A)$

e.) $f^{-1}([1, 4])$

Theorem 4. *Suppose $f\colon A \to B$ is a function, and let U and V be subsets of A. Then ...*

Proof. Suppose $x \in U \cup V$. Then $x \in U$ or $x \in V$.$^{(a)}$ If $x \in U$, then $f(x) \in f(U)$,$^{(b)}$ and so $f(x) \in f(U) \cup f(V)$.$^{(c)}$ Otherwise if $x \in V$, then $f(x) \in f(V)$,$^{(d)}$ so $f(x) \in f(U) \cup f(V)$.$^{(e)}$ In either case $f(x) \in f(U) \cap f(V)$. Suppose $f(x) \in f(U) \cup f(V)$. Then $f(x) \in f(U)$ or $f(x) \in f(V)$.$^{(f)}$ If $f(x) \in f(U)$, then $x \in U$.$^{(g)}$, and if $f(x) \in f(V)$, then $x \in U$.$^{(h)}$. In either case, $f(x) \in U \cup V$, completing the proof. □

⋆ **Exercise 5.**

(1) Complete the statement of Theorem 4.
(2) There are a few errors (typos?) in the proof. Find and correct them.
(3) Provide a few words of justification for each line (e.g. cite the appropriate definition).
    (a)
    (b)
    (c)
    (d)
    (e)
    (f)
    (g)
    (h)

⋆ **Exercise 6.** Take the statement in Theorem 4, and replace all ∪'s with ∩'s and ∩'s with ∪'s. Then do the same in the (corrected) proof. For each new statement (a)–(h), provide the appropriate justification. If the statement is false, explain why.

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)