MATH 2001

Definition. Let A be a set. A *basis for a topology* on A is a collection $\mathscr{B}$ of subsets of A (called *basis elements*) that satisfies the following.

(1) For each $x \in A$, there exists a $B \in \mathscr{B}$ such that $x \in B$.
(2) If B_1 and B_2 are any two elements in $\mathscr{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathscr{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

$\star$ **Exercise.** In each of the following cases, prove that $\mathscr{B}$ is a basis for a topology on A, or explain why it is not.

a) $A = \{a, b, c, d, e\}$, and $\mathscr{B} = \{\{a, b\}, \{c, d\}, \{c, d, e\}\}$.

b) A is a set, and $\mathscr{B} = \mathscr{P}(A)$.

Proof. We prove that $\mathscr{P}(A)$ satisfies the definition to be a basis for a topology on A.

Note that for each $x \in A$, we have $x \in \{x\}$ and $\{x\} \in \mathscr{P}(A)$, thus satisfying the first condition of the definition.

Moreover, for any B_1 and B_2 in $\mathscr{P}(A)$, setting $B_3 = B_1 \cap B_2$ satisfies the second part of the definition, as $B_3 \in \mathscr{P}(A)$, $B_3 \subseteq B_1 \cap B_2$, and $x \in B_1 \cap B_2 \Rightarrow x \in B_3$. $\square$

c) A is a set, and $\mathscr{B} = \{X \in \mathscr{P}(A) : |X| = 3\}$.

Proof. The set $\mathscr{B}$ is not necessarily a basis for a topology on A.

If $1 \leq |A| \leq 2$, then it is impossible to satisfy the first criterion in the definition as $\mathscr{B}$ is empty.

If $|A| > 3$, then $\mathscr{B}$ is not a basis for a topology on A as it violates the second part of the definition. To see why the second condition cannot be satisfied, let w, x, y, and z be distinct elements of A. Then $\{x, y, z\}$ and $\{w, x, y\}$ are elements of $\mathscr{B}$, but $\{x, y, z\} \cap \{w, x, y\} = \{x, y\}$ is a set of cardinality 2, and thus B does not contain an element that is a subset of $\{x, y\}$ (every element of $\mathscr{B}$ is a set of cardinality 3). $\square$

d) $A = \mathbb{R}$, and $\mathscr{B} = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$. (The interval (a, b) is the set $\{x \in \mathbb{R} : a < x < b\}$.)

e) $A = \mathbb{R}$, and $\mathscr{B} = \{[a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\}$. (The interval $[a, b)$ is the set $\{x \in \mathbb{R} : a \leq x < b\}$.)

f) $A = \mathbb{R}$, and $\mathscr{B} = \{[a, b] : a, b \in \mathbb{R}\}$. (The interval $[a, b]$ is the set $\{x \in \mathbb{R} : a \leq x \leq b\}$.)

g) $A = \mathbb{R}$, and $\mathscr{B} = \{(a - 1, a + 1) \subset \mathbb{R} : a \in \mathbb{R}\}$.

h) $A = \mathbb{R}$, and $\mathscr{B} = \left\{\overline{\{x\}} : x \in \mathbb{R}\right\}$.

i) $A = \mathbb{R}$, and $\mathscr{B} = \left\{\overline{C} : C \subset \mathbb{R}, C \text{ is a finite set}\right\}$. (That is, $B \in \mathscr{B}$ if $B = \mathbb{R} - C$ for some finite set C.)

Definition. Let A be a set, let $\mathscr{B}$ be a basis for a topology on A, and let X be a subset of A. The set X is *open* if for each $x \in X$, there exists a $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq X$.

Definition. Let $\mathscr{B}$ be a basis for a topology on A, and let X be a subset of A. The set X is *closed* if $A - X$ (the complement of X in A) is open.

$\star$ **Exercise.** For each set $\mathscr{B}$, determine which the following subset of $\mathbb{R}$ are open and which are closed with respect to the basis $\mathscr{B}$. (Assume $a, b \in \mathbb{R}$.)

I. (a, b) II. $(a, b]$ III. $[a, b)$ IV. $[a, b]$ V. $\mathbb{Z}$ VI. $\mathbb{R}$ VII. $\varnothing$

(1) $\mathscr{B} = \{(x - \epsilon, x + \epsilon) : x \in \mathbb{R}, \epsilon \in \mathbb{R}, \epsilon > 0\}$
(2) $\mathscr{B} = \{[x, x + \epsilon) : x \in \mathbb{R}, \epsilon \in \mathbb{R}, \epsilon > 0\}$
(3) $\mathscr{B} = \{(x - \epsilon, x] : x \in \mathbb{R}, \epsilon \in \mathbb{R}, \epsilon > 0\}$
(4) $\mathscr{B} = \{[x - \epsilon, x + \epsilon] : x \in \mathbb{R}, \epsilon \in \mathbb{R}, \epsilon > 0\}$
(5) $\mathscr{B} = \{\overline{C} : C \subset \mathbb{R}, C \text{ is a finite set}\}$

Since closed sets in $\mathbb{R}$ are defined in terms of complements, the following result may be useful. Some of you proved these statements on the last assignment, but regardless if you have proved these statements or not, you may now use them without proof.

Theorem (DeMorgan's Laws (for sets))**.**
(1) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
(2) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Homework 1. Due Thursday, October 1, 6PM.

(1) Prove that the union of any number of open sets is open.

Proof. (Done in class.) Let $\{A_i : i \in \Lambda\}$ be a collection of open sets (i.e. each A_i is an open set), and let A be the union of all the sets in this collection. That is

$$A = \bigcup_{i \in \Lambda} A_i$$

To prove that A is open, we show that A satisfies the definition of an open set. That is, we show that for each $x \in A$, there exists a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq A$.

Suppose $x \in A$. Since $x \in \bigcup_{i \in \Lambda} A_i$, it follows from the definition of union that $x \in A_i$ for some i. Then since A_i is open, there exists a $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq A_i$. Moreover, since $B \subseteq A_i$ and $A_i \subseteq A$, it follows that $B \subseteq A$, concluding the proof. $\square$

(2) Prove that the intersection of two open sets is open.
(3) Prove that the union of two closed sets in $\mathbb{R}$ is closed.
(4) Prove that the intersection of two closed sets in $\mathbb{R}$ is closed.