MATH 2001
FINAL EXAM

1. (a) Write out the first seven rows of Pascal's triangle (up to the row starting 1, 6).

   (b) Define what it means for a to *divide* b. Include any necessary conditions for a and b in your statement.

   (c) Consider the following definitions.

   Definition. If a divides b, then we say that a is a *divisor* of b.

   Definition. A number $p \in \mathbb{N}$ is *prime* if $p > 1$ and the only positive divisors of p are 1 and p.

   Prove that if p is prime, then p divides $\binom{p}{k}$ for all $0 < k < p$.

(d) According to the *binomial theorem*,

$$(x+y)^n = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad .$$

(e) Explain why $(x+y)^n = x^n + y^n$ is not a true statement in general. (A single counterexample would be sufficient.)

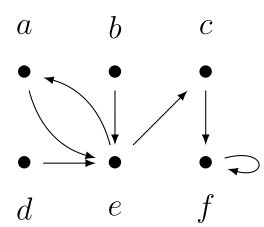
(f) Define what it means for a to be *congruent* to b modulo n. As before, include any necessary conditions on a, b, and n in your statement.

(g) Prove that if p is a prime number, then $(x+y)^p \equiv x^p + y^p \pmod{p}$.

2. (a) Define what it means for R to be an *equivalence relation* on A.

(b) Let $A = \{a, b, c, d, e, f\}$, and let R be the relation described by the following graph.

What is the minimal number of elements that need to be included for R to be transitive relation? List these elements in set notation.



(c) What is $\mathbb{R}^2$? (Given your answer in set-builder notation, or give a description using vocabulary that is relevant to this course.)

(d) Let $A = \mathbb{R}^2 - \{(0,0)\}$ (all of the elements of $\mathbb{R}^2$ except for $(0,0)$). Let R be the relation on A that is defined as follows: $((x,y),(z,w)) \in R$ if and only if there exists a non-zero number $\lambda \in \mathbb{R}$ such that $(x,y) = (\lambda z, \lambda w)$. (In other words, $x = \lambda z$ and $y = \lambda w$.)

Prove that R is an equivalence relation on A.

3. (a) Define what it means for A to be a *finite set*.

(b) Let $\{A_i : i \in \mathbb{N}\}$ be a collection of finite sets (i.e. A_i is finite for each $i \in \mathbb{N}$).

Let $B_n = \bigcup_{i=1}^{n} A_i$. Give a proof by induction that B_n is a finite set for each $n \in \mathbb{N}$.

(c) Give an explicit example for when $\bigcup_{i=1}^{\infty} A_i$ is finite.

Give a second example where $\bigcup_{i=1}^{\infty} A_i$ is infinite.

(d) Prove that $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$. (Here, A^c denotes the complement of A.)

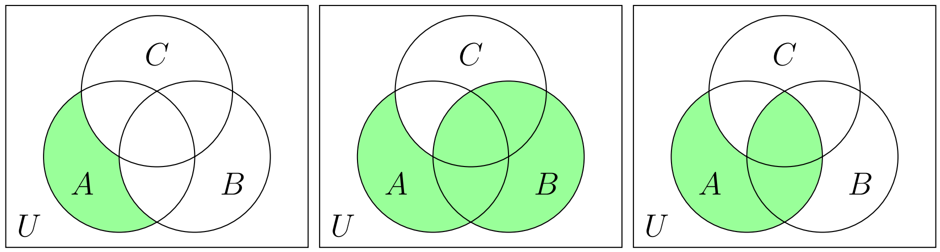
4. Join each set to its corresponding diagram. If no diagram exists, leave that statement unconnected.

   (a) $(A \cap A) - (C \cup B)$

   (b) $((A \cap A) - C) \cup B$

   (c) $A \cap ((A - C) \cup B)$

   (d) $A \cap (A - (C \cup B))$



5. Let A and B be sets, and let f be a function from A to B. Let $X \subseteq A$ and $Y \subseteq B$. Consider the following proof.

Proof. Suppose that $V, Y \subseteq B, x \in f^{-1}(V)$, but $x \notin f^{-1}(Y)$. Then $f(x) \in f(V)$. Then $f(x) \in f(Y)$. So $x \in f^{-1}(Y)$, which is impossible, completing the proof. $\square$

(i) What is the writer attempting to prove, and what is the style of proof being used?

(ii) What is begin used to justify the second sentence? Write out the complete definition.

(iii) There are some problems with the third sentence. What is the issue, and how can the proof be corrected so that the third sentence follows from the second?

(iv) The fourth sentence also has an error. What is the issue here?

(v) We would like to correct the fourth statement if possible. Suppose $x \notin f^{-1}(V)$ (as the author assumes). What can we say about $f(x)$?

(vi) Using the suggestions from (ii)–(v), rewrite the proof so that it gives a valid argument for the statement in (i).