Discrete Mathematics Course Material (Math 2001)
Spring 2016
University of Colorado Boulder

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May 12, 2016

Contents

Preface

This document is a compilation of all material related to my Discrete Math course in Spring 2016, as well as some thoughts as to my philosophy regarding this course. The result was this behemoth.

I put this document together so that anyone interested in viewing my materials could do so easily without having to click though dozens of files. Most items are accompanied by a short summary and a reflection of sorts (e.g. how the material should be change before being reused).

Feel free to use anything in here for any reasonable purpose without my permission.

Chapter 1

Course summary

1.1 Philosophy and objectives

There are many variations and visions of what this course should be, but I hope there is little disagreement that this course serves as the foundation for mathematical writing and literacy. By the end of the course, I consider the following to be a minimum set of expectations.

1. Understand what it means to apply and/or satisfy a mathematical definition.
2. Understand how to manipulate logical statements.
3. Develop a familiarity with the standard types of proof, and have the ability to set up the initial statements for each type of argument.
4. Know the standard for a rigorous proof, and have the ability to write simple arguments in line with these qualifications.
5. Develop the capablility of reading and understanding formal arguments that are not one's own.

I also expect that my students develop the ability to use LaTeX proficiently.

Of these five or six points, I do not have any other specific objectives for the course. In particular, I do not have any goals which are content oriented. There is always a trade off between spending time refining techniques and developing content, and I would place my methods squarely in the former camp. My class was (and is) a fairly cut-and-dry, 'here is a definition, now now write a proof' course. In terms of content, I would not be surprised if I covered the least of any of the 2001 sections this semester.

Just to give an overview of the contents of my course, I covered material corresponding to the following sections of Richard Hammack's *Book of Proof*. In chronological order:

Chapter 1	Sections 1–8	(sets and set operations)
Chapter 8	Sections 2, 3	(proofs involving subsets and set equality)
Chapter 2	Sections 2, 3, 7, 10	(logical statements)

Chapter 5	Section 1	(proof by contrapositive)
Chapter 6	Section 1	(proof by contradiction)
Topology	Munkres' Chapters 12 and 13	(basis of topology, open and closed sets)
Chapter 11	Sections 1–5	(relations, equivalence relations)
Chapter 12	Section 1, 2, 5, 6	(functions and inverses)
Chapter 10	Section 1	(proof by induction).

To reiterate, I treated the book as a supplement to the class. I did not cover the exact contents of these sections, and the material that I did cover likely touches on more sections than I have listed here.

It would not be an exaggeration to call my course 'statements about sets'. I freely admit that I eschewed anything computational, including many topics from number theory and combinatorics that are often consider standard for a course like this. I rejected these topics in order to deny my students the foothold of intuition, "common knowledge," and algebraic...nonsense...that too often infiltrates proofs at this level.

Is it unreasonable for me to deny the familiar? Is it cruel to hang my students on the thread that is the definition of a topology? Perhaps it is extreme, but this is my approach to addressing the common fallacies in proof writing. First we learn to argue devoid of context. Then when we return to the familiar, we have the wherewithal to avoid the pitfalls.

Now, do I follow what I preach? Not exactly. I do not avoid examples in $\mathbb{R}$, $\mathbb{R}^2$, or subsets thereof. Still, I think my framework is consistent. We start with the basic about sets and gradually expand the complexity of the statements. In the end, all the arguments are anchored in the familiar (albeit tedious) argument that X is a subset of Y.

That is not to say that I would not teach beyond the 'complexity' of sets. I just do not feel that I can do any particular subject—combinatorics, probability, graph theory, number theory—justice by piddling around in each subject for a few weeks at a time. I would happily teach a foundational course based on graph theory or combinatorics or...well, just give me my students for three years and I will give each subject the justice and flavor it deserves.

1.2 Assignments and assessment

1.2.1 Worksheets

All the worksheets for the course can be found in the `Worksheets` folder.

I use the worksheet partially as guided notes, and partially as a mechanism for self-guided learning.

Just to give an idea of my average class period: I give a brief introduction to any new concepts, give a few examples, and clarify instructions for the problems on the worksheets.

This generally takes between 15 and 25 minutes depending on the material. I then have the students work on the remainder of the problems. By the end of the first week, students consider group work to be standard, so they naturally help each other out and compare solutions.

In the mean time, I walk around, answer questions, give clarification, and check answers. If the class is struggling on a particular problem, then we might do the problem together as a class. I might end the class or start the class following a worksheet by going through the solutions, but I do not do that for every problem, much less every worksheet. Sometimes I have the students put the solutions on the board, and disagreement is always a nice catalyst for discussion.

1.2.2 Proof portfolio

The template for the Proof Portfolio is the `ProofTemplate.tex` file in the `ProofPortfolio` folder.

These were the primary homework assignments. One new proof every two to three class periods. Students maintained all of their proofs in a single online document, which I had access to. Following each due date, I would go into each file and write comments, and then the students would have the opportunity to rewrite their proof. At the beginning of the semester, the students had two chances to rewrite their arguments, and later in the semester, that was cut down to once. Effectively, each assignment had 'draft' deadlines and 'final' deadlines. A few students chose to only submit 'final' drafts, which was okay by me.

The positives of these assignments (assuming that the student submitted work on time) were that students received a lot of direct feedback from me. By having everything tex-ed and online, it was very easy for me to jump between documents, as well as review the comments I had written on previous iterations of the proofs. And since the students had to respond to my comments, my time was not spent in vain. This also made it easy for students to review their arguments (just copy-paste and fix the errors).

On the down side, the volume of grading was often overwhelming to the point that it was near impossible to keep up. In part, my comments we often so extensive that I might spend half an hour or more commenting on a single proof. This is, of course, an absurd waste of time on my part, and I would be better served calling the student in to office hours. The time commitment to each assignment also meant that I did not necessarily assess a great variety of problems. That being said, I would do this all again.

I had intended do to more peer review in this class—that would have cut down on some of the grading perhaps—but I never found time to organize those sessions.

For extreme versions of the comments I might write on a proof, or if you are interested in how I wrote comments in the files, see Quiz 8 and 9 later in this document.

1.2.3 Book exercises

The template for book exercises is `TexTemplate.tex`, which is in the `Syllabus` folder.

I assigned very few of these during the semester despite claiming to assign these 'daily' in the syllabus. Book problems were only graded for completion. I assigned book exercises generally to reinforce new definitions, and at the beginning of the year, these were served as a 'get used to writing math in latex' exercise.

1.2.4 Quizzes

All quiz materials are in the `Quizzes` folder.

Quizzes were given approximately once per week, and I give a more detailed explanation of the quizzes in the introduction to that chapter below. This was the chief means for assessing whether the students were keeping up with the terminology. Most quizzes have a 'state the definition of [...]' section. But outside of that, the types of questions varied considerably.

1.2.5 Final exam

Final exams are also in the `Quizzes` folder.

This was the only exam in the course, and it is lightly weighted. I cared more about having the students develop their writing skills and otherwise staying on top of the content as it is presented. Exams just do not seem to fit my view of the course. I much prefer to give many small assessments rather than a few big ones.

1.3 Supplemental material

1.3.1 Flashcards

I made these using the `flashcard` package. Very easy to use, and very useful. The flashcards contain (nearly?) all the relevant terms for this course.

I also included a fair bit of vocabulary for graph theory. They do not show up in the pdf, but you can find them by scrolling to the end of the tex file.

1.3.2 LaTeX guide

I wrote this document as a searchable guide to everything that might be relevant to this course—and more. Most everything comes with examples, and the symbols come with English translations.

Math 2001 — Discrete Mathematics

SPRING 2016 SYLLABUS

Class Location: MUEN E417, MWF 2:00–2:50 PM
Instructor: T. Alden Gassert
Office: MATH 223
Email: Thomas.Gassert@colorado.edu
Office Hours: M 3:00–4:30 PM, Tu 9:00–10:00 AM, F 3:00–4:00 PM, and by appointment.
Text: *Book of Proof* by Richard Hammack. A `pdf` of the book is available at his website:
http://www.people.vcu.edu/~rhammack/BookOfProof/BookOfProof.pdf
Paperback copies are also available for purchase at the bookstore (∼\$15).

About the course: There are three main themes to this course: developing mathematical literacy, understanding discrete mathematical objects, and training effective thinking.

As in any subject, the sharing of ideas depends on our ability to convey and comprehend concepts in an effective manner; mathematics is no exception. We need a vocabulary (definitions) to express our ideas (theorems), and we must be able to explain why our ideas are valid (proof). This structure should be familiar, however in this course, we will make these arguments in accordance with the rules of the English language. That is, our ideas and explanations will be written and spoken in full English sentences. As you progress as a mathematician, you will find that mathematics has less to do with solving equations (as you have been doing in virtually every math class up till now) and more to do with the exploration of conceptual ideas, thus necessitating the use of the English language.

The mathematical content of this course are objects and concepts that apply broadly to many areas of mathematics. Some of these topics include sets, relations, (injective, surjective, and bijective) functions, and modular arithmetic. For the most part, our sets are discrete (loosely, this means we can count the items in the set), and our relations and functions will be maps between discrete sets. (Hence the name of the course.)

Finally, and perhaps most importantly, is training ourselves to think effectively. What does this mean? Broadly, this means accepting challenges and having the courage to fail. It means learning from mistakes. How can we gain the most out of our mistakes? On one hand, this involves understanding why we failed. On the other, we need to fail in a way that provides some meaningful information, and that begins with asking the right questions. How do we ask right questions? That is entirely experiential. There is no innate knowledge that points us in the correct direction. So be bold and question.

Course goals:

- The primary goal is to develop a foundation of mathematical literacy. By the end of the course, I expect that you have a command of the standard styles of proof, and that you write with proper mathematical (and English) syntax and grammar.
- Moreover, I expect you to read and understand the proofs of your peers, or find counter-arguments if the proofs are incomplete.

Assessment:

10% Exercises — assigned daily

10% Class participation — class discussion, in-class assignments, group assignments

30% Quizzes — at least one per week, generally short and at the beginning of class

30% Proof portfolio — your writing will be tracked over the course of the semester

20% Final Exam

Academic integrity: Learning is a collaborative experience, however all work must be your own (or your group's if it is a group assignment). Violations to the academic honor code will be penalized with a minimum of a 20 point deduction to your final grade subject to the finding of the Honor Code Office.

Special accommodations, religious observances, etc.: If you qualify for special accommodations due to a disability, you will need to provide to me a letter from Disability Services. Please attend to this matter in a timely fashion.

Please inform me as soon as possible if you will miss an exam or a homework assignment due to religious observance so that we have time to arrange a reasonable accommodation.

Chapter 2

First week sheets

First week things are available in the `FirstDays` folder.

I feel that the first week of the course is significant enough to warrant its own section in this document. In my younger years, I considered the first day to be a zero content day—and I think many students have the same impression. I used to spend time going over a syllabus, outlining the overarching ideas of the course, and offering a history or perspective on the material...

These days, the first week is all about setting the precedents for the course. In the first week of the course this semester, the first day set up mechanisms for class participation and group work, the second was dedicated to LaTeX, and the third to peer review.

2.1 First day

My first day activities generally proceed as follows.

1. Syllabus. Contact information and a brief overview of expectations for the course (e.g. how often and what type of assessments). Generally I set office hours based on the schedules of my students, so I survey when students would most likely come to the office. This runs no more than 10 minutes. The survey can be done online after class.

2. Introductions. I ask the students to turn to those next to them and introduce themselves. Sometimes I give prompts: What is your major? Where are you from? What was your last math class? Generally I participate as well. If anyone is not talking, then I will introduce myself to them, get them chatting, and then facilitate introductions with those nearby. I generally let this run for at least five minutes. For me, there are always more students to meet—again, always looking to engage those who are not actively conversing. I set the precedent, so if I am continuing a conversation, so will the rest of the class. For this, the more chatter the better.

3. Introductions II. Now the twist. After bringing the class back to order, each student

must introduce one other student by giving that student's name and a few facts. No student is permitted to introduce themselves. Sometimes I start things off by introducing one of the students I talked to (and then they have to introduce me). This activity sets the president that each student should expect to speak in front of the class. It is also a subtle thing, but I always try to move to the opposite side of the room as the speaker. S/he naturally looks in my direction, so I position myself so as to have the student speak towards the majority of the class.

4. Group project. I ask students to form groups, which happens naturally given the previous ice-breaker. I assign a logic problem to each group to work on for the remainder of the class. I generally assign a variety of problems, which each group getting something different. You can find the projects in the `FirstDays` folder.

2.2 Second day

As mentioned previously, one of the requirements for my classes is that all work (other than quizzes and the final exam) is written in LaTeX, and I use the second day to give the introduction. In preparation, I ask my students to bring their laptops to class. I have a worksheet prepared (Worksheet 0 in the next Chapter) and a template available to get the students off on the right foot (though I cannot say that went entirely smoothly this year). The program that I have my students use is Overleaf (`Overleaf.com`), which is a free, online, collaborative, tex editor.

The first assignment of the course is to write up the assignment from the first day in tex. The primary goal is to get the students to fully explain their problem and solution. Together, the worksheet and this first assignment offers a gentle introduction as to how to write in the editor and how things show up in the pdf.

The instructions for Assignment 1 are included at the end of this chapter. The assignment contains many parts, but this is because the groups were large and I wanted to give enough so that each person would have to write a portion of the document.

2.3 Third day

By having the students tex their work, it is very easy for me to alter their documents to use for classroom activities. On this day, I printed off each group's solution (just part 2 of the first assignment). Each group was then assigned to review one other group's write up. Instructions are found on the Assignment 2 sheet (attached below).

Part of the assignment includes having the groups write their own rubric. The idea is not only for the students to crystalize the standard to which they are holding the other group, but also to document the standard to which they must hold themselves in their own writing.

MATH 2001
ASSIGNMENT 1
DUE FRIDAY, JANUARY 15 AT 1PM

As a group, please write up the solution to your problem in a new Overleaf document, and share the file with me when you are finished. In your write up, please address the following points.

(1) In your own words, describe the problem that your group was given. As you write this up, consider any difficulties or confusions you encountered when first reading/working on the problem.

(2) Explain your solution (or solutions!) to the problem. Even if you end up with a single answer, your group may have arrived at this answer from difference approaches. Explain each of these lines of reasoning.

(3) If your group has not solved the problem, write up your thought process: what did you try, and what issues did you run into?

(4) Explain a line of reasoning that you have not yet explored that could potentially lead to a better solution, or provide a counter argument to an approach that someone else might try. (For the counter argument, this might be something your tried before realized the approach was not valid.)

(5) How might you generalize your problem? In other words, there is nothing all that special about the numbers in your problem, so what happens when we change the numbers? Provide a generalization of your problem and explain how your solution might change. You do not have to write out a complete solution to the new problem; just explain whether the method you used to solve the original problem would still apply, and provide a guess as to what the solution to the more general problem would be.

Date: January 15, 2016.

MATH 2001
ASSIGNMENT 2

Designate two people in your group to be scribes. At the end of class, you will turn in two marked up copies: one will go to me, and the other will go to the group that wrote up the problem.

Instructions.

1.) Write a one paragraph summary of the problem and solution (or approach if the solution is incomplete). You do not need to go into great detail, just provide an overview of the approach that was used to solve the problem.

2.) On a scale of 0–5, how confident are you in the validity of the solution (with 0 being no confidence, and 5 begin absolutely confident)? Write one or two sentences to explain your level of confidence. For example, you might write 'the argument is clear, precise, and easy to follow' for a 5, or 'the argument does not address the original question' for a 0. (Note, this is not a grade. A poorly written argument might still receive a 5.)

3.) Mark up the document. Which parts are well written? Which parts could the authors improve, and how might those sections be improved? (E.g. is there anything you find confusing, and why are you confused?) Mark any spelling/punctuation/grammar/etc errors as well. Feel free to comment on formatting as well.

4.) Finally, if you were going to give this assignment a grade, how would you score it? (The scoring system is completely up to you.) On the last page, identify the criteria by which you are assigning your grade. Your criteria might be very general (logic) or picky (punctuation). For each qualification, give a one line explanation of how the document met your expectation.

    (For example, maybe you assign one point to punctuation and spelling, and you consider fewer than two errors to be acceptable. However, the document has four of these errors, so you assigned 0 points in this category.)

5.) Put your names on the copy that you are turning into me. (You do not need to put your names on the other copy.

Follow up. Over the weekend, write up a final version of the assignment taking your peers' comments into consideration. Send me your Overleaf link before our next class.

Date: January 15, 2016.

Category	What are your standards, and how has the document met them?

Chapter 3

Worksheets

All worksheets are found in the `Worksheets` folder.

3.0 Introduction to LaTeX

It was a requirement in my course that all assignments be tex-ed. For the entirety of the course, the students maintained two tex documents. One containing all problems assigned from the book, and the other containing a particular set of proofs. I provided templates for each of these documents. The book problem template is `TexTemplate.tex`, found in the `Syllabus` folder. The template for the proofs is `ProofTemplate.tex`, found in the `ProofPortfolio` folder. I also provided a LaTeX guide (see Appendix B).

The intension of this first worksheet is to give an introduction to latex. I had the students bring their computers to class, then attempt to reproduce the worksheet in a tex editor (Overleaf).

I had issues getting the students started with opening a new Overleaf document with a template I provide. Most straightforward solution I can think of for the future is to start an Overleaf document, then just have the students copy the tex into a new Overleaf document.

This is may not the most effective way to teach tex-ing, but it seemed to work for my class.

As mentioned in the previous chapter, the first assignment in the course was to write up a logic problem in Overleaf. Most of these problems involved very little math, and the primary focus was to give a gentle introduction to writing in Overleaf, and secondly to set the tone for the course: arguments should be written formally and completely.

LATEX WORKSHEET

ALDEN

LaTeX is a type-setting program created by the famous computer scientist, Donald Knuth. Unlike word processors, LaTeX documents are fully customizable... you just have to tell the complier exactly what you want to have happen. Today, LaTeX is used widely in math and the sciences. Scientific publications, including many of your textbooks, were written using this program.

The goal today is for you to reproduce this worksheet. To make a new paragraph, hit the return key twice. We are about to create a numbered list, so to do that, type the following into your .tex file:

```
\begin{enumerate}
\item Edit the title and author fields in your .tex file.
\end{enumerate}
```

(1) Edit the title and author fields in the .tex file.

(2) Add a new numbered item using the “item” command. Use the left-quote (on the tilde key) to get your quotation mark going the correct direction.

(3) LaTeX has a “math mode” that is triggered by the dollar sign. Write $y = mx + b$ by putting the entire expression “y=mx+b” between dollar signs.

(4) Use the caret for super scripts: $x^2 + y^2 = 1$. (Don't forget math mode!)

(5) The underscore is used to make subscripts: $ax_1 + bx_2 = x_3$.

(6) Some symbols are special and need to be escaped with a backslash (meaning you need to type a backslash before each of these symbols): { } # $ % &. To get the backslash to appear in text, type \textbackslash.

(7) For inequalities, use \le and \ge. Otherwise, use the symbol on the period and comma keys: $a \le b < c$ and $7 > 6 \ge 1$.

(8) Type \bbn, \bbz, \bbq, \bbr for $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

(9) For fractions, use \frac{a}{b}: $\frac{13}{4}$ is in $\mathbb{Q}$.

(10) \binom{a}{b} is used for binomial coefficients: $\binom{n}{k}$.

(11) Sometimes, we want to separate the math from text. Type the following:

```
\begin{align*}
(x+y)^2 &= (x+y)(x+y)\\
&=x^2 + 2xy + y^2.
\end{align*}
```

The \\ starts a new line, and the & sets the alignment.

(12) Type the following:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Date: May 5, 2016.

3.1 Sets

Section 1.1 of Hammack.

MATH 2001
INTRODUCTION TO SETS

(1) Consider the following five items:

I) $\varnothing$ II) $\mathbb{Q}$ III) $\{\varnothing\}$ IV) π V) $\{\{1\}\}$

Give a brief explanation as to why each item is or is not a set.

(2) Consider the following sets:

I. $\varnothing$ II. $\{\{\}\}$ III. $\{\varnothing, \{\}\}$

How would you read each of these items out loud (write out what you would say)? Which of these is/are equal to $\{\varnothing\}$?

(3) True or false? Explain.

$$\{\{c, a\}, b, \{a\}\} = \{\{a, c\}, \{b\}\}$$

(4) Consider the following set:

$$A = \{\{\{x\}, d\}, \{d, x\}, \{x\}, \{d, \{x\}\}, \{\varnothing, x\}\}.$$

(a) Which of the following statements are true?

(i) $x \in A$

(ii) $d \notin A$

(iii) $\{x, d\} \in A$

(iv) $\varnothing \in A$

(b) What is the cardinality of this set?

3.2 Subsets

Section 1.3 and 1.4 of Hammack. I have skipped over Section 1.2 at the moment. The Cartesian product is a binary operation on sets, so I placed it along side the other binary operations. In the mean time, subsets are sets in sets—why would you not follow sets with sets in sets?

MATH 2001
INTRODUCTION TO SUBSETS

Exercise 1. What is a *subset*? State the exact definition.

Exercise 2. Let A be a set. In your own words, what is the difference between an element of A and a subset of A?

Exercise 3. Let $A = \{1, 2, \{3\}, \{2, 1\}\}$. True (**T**) or false (**F**)?

T	**F**	:	$\varnothing \in A$	**T**	**F**	:	$\varnothing \subseteq A$
T	**F**	:	$3 \in A$	**T**	**F**	:	$3 \subseteq A$
T	**F**	:	$\{1, 2\} \in A$	**T**	**F**	:	$\{1, 2\} \subseteq A$

Are your answers here justified by the statement you gave in Exercise 2? Edit your statement if necessary.

Definition. Let A be a set. The *power set* of A is the set of all subsets of A.

Notation. The power set of A is denoted by $\mathscr{P}(A)$ (`\mathscr P(A)`). In set builder notation,

$$\mathscr{P}(A) = \{X : X \subseteq A\}.$$

"The power set of A is the set of all X, where X is a subset of A."

Exercise 4.

i.) Let $A_0 = \{\}$ be the empty set. Write the set $\mathscr{P}(A_0)$ explicitly. (Remember, $\mathscr{P}(A_0)$ is a set, so use the appropriate set notation.)

ii.) Let $A_1 = \{a\}$. What is $\mathscr{P}(A_1)$?

iii.) Let $A_2 = \{a, b\}$. What is $\mathscr{P}(A_2)$?

iv.) Let A_n be a set of cardinality n. Make a guess as to the cardinality of $\mathscr{P}(A_n)$. In a few sentences, explain how you came about your answer. If you can prove your claim, even better.

Exercise 5. Let A be a set. True (**T**) or false (**F**)?

T	**F**	: $\varnothing \in \mathscr{P}(A)$		**T**	**F**	: $\varnothing \subseteq \mathscr{P}(A)$
T	**F**	: $A \in \mathscr{P}(A)$		**T**	**F**	: $A \subseteq \mathscr{P}(A)$
T	**F**	: $\mathscr{P}(A) \in \mathscr{P}(A)$		**T**	**F**	: $\mathscr{P}(A) \subseteq \mathscr{P}(A)$

Homework. Due Wednesday, January 25 at 2pm.

- Read Sections 1.3 and 1.4 from the text.
- Complete the following exercises (add these to your Overleaf file with the other book problems).
  - Section 1.3: 2, 3, 11, 12.
  - Section 1.4: 5, 14, 17.
  - From this worksheet: Formalize your thoughts from Exercise 4.iv, and write a one paragraph explanation of your guess for the cardinality of $\mathscr{P}(A_n)$.

3.3 Cartesian product

Section 1.2 of Hammack. Standard stuff.

MATH 2001
INTRODUCTION TO CARTESIAN PRODUCT

Exercise 1. What is a *Cartesian product*? State the exact definition.

Exercise 2. Let $A = \{0, 1\}$ and $B = \{\varnothing\}$. Write out the sets A^2 and $B \times A$ explicitly.

Exercise 3. Let $A = \{a, b, \{a, b\}, (a, b), \mathbb{Z}\}$. True or false?

T	**F** :	$\varnothing \in A$		**T**	**F** :	$\varnothing \subseteq A$
T	**F** :	$\varnothing \in \mathscr{P}(A)$		**T**	**F** :	$\varnothing \subseteq \mathscr{P}(A)$
T	**F** :	$\{a, b\} \in A$		**T**	**F** :	$\{a, b\} \subseteq A$
T	**F** :	$\{a, b\} \in \mathscr{P}(A)$		**T**	**F** :	$\{a, b\} \subseteq \mathscr{P}(A)$
T	**F** :	$(a, b) \in A^2$		**T**	**F** :	$(a, b) \subseteq A^2$
T	**F** :	$\{(a, a)\} \in A^2$		**T**	**F** :	$\{(a, a)\} \subseteq A^2$
T	**F** :	$\{a, b\} \in A^2$		**T**	**F** :	$\{a, b\} \subseteq A^2$
T	**F** :	$1 \in A$		**T**	**F** :	$1 \subseteq A$

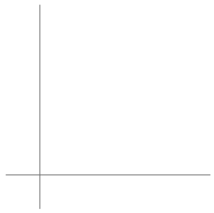
Exercise 4. Let $A = \{0, 1\}$, $B = \{\varnothing\}$, and $C = A \times B$. Consider the following sets:

- $A \times A \times B$
- $A \times C$
- $A^2 \times B$
- $A \times (A \times B)$
- $(A \times A) \times B$

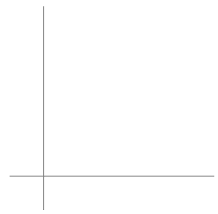
What are the distinctions between these sets? Are all/any of them the same? If not, how do the sets differ?

Exercise 5. Sketch each of the following sets in the x, y-plane.

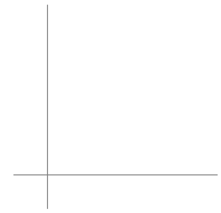
a. $[1,2]^2$



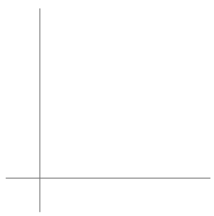
c. $[1,2] \times \{1,2\}$



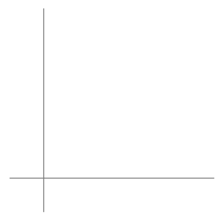
e. $\mathbb{R} \times \{1,2\}$



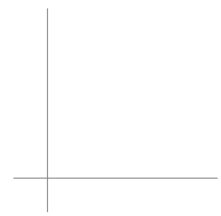
b. $\{1,2\}^2$



d. $\mathbb{R} \times \mathbb{N}$



f. $(0,1) \times \mathbb{R}$



Exercise 6. Suppose A and B are finite sets. What is the cardinality of $A \times B$? In a few sentences, justify your claim.

Homework. Due Friday, January 29 at 2pm.

- Read Section 1.2 from the text.
- Complete the following exercises (add these to your Overleaf file with the other book problems).
  - Section 1.2: 2a, 2b, 6

3.4 Set operations

Sections 1.5–1.7 of Hammack. Some questions are challenging without yet discussing how to manipulate logical statements (in particular 5.iv stands out), but the point is to just have the students write something reasonable.

MATH 2001
SET OPERATIONS

Exercise 1. Express each of the following sets in set builder notation.

i.) $A \cap B =$

ii.) $A \cup B =$

iii.) $A - B =$

iv.) $\overline{A} =$

Exercise 2. Let $A = \{\text{NJ, MD, MA, ME, CO}\}$, $B = \{\text{MA, ME, CO, UT}\}$, and U be the set of states in the US. Give an explicit answer to each of the following.

i.) $A \cap B =$

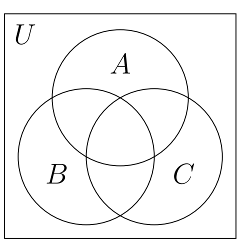
ii.) $A \cup B =$

iii.) $A - B =$

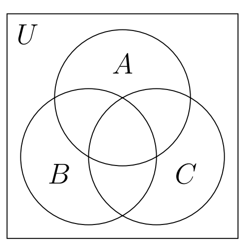
iv.) $|\overline{A}| =$

Exercise 3. Shade the region in the Venn diagram that corresponds to each of the following sets.

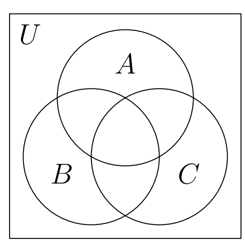
i.) $A \cap B$



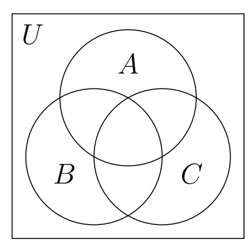
iii.) $\overline{A} \cap C$



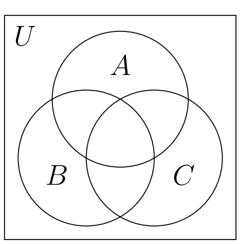
v.) $(A \cap A) - (C \cup B)$



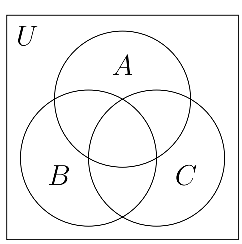
vii.) $A \cap ((A - C) \cup B)$



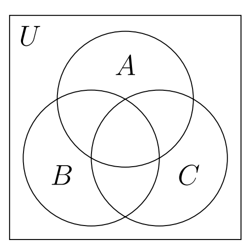
ii.) $B \cup C$



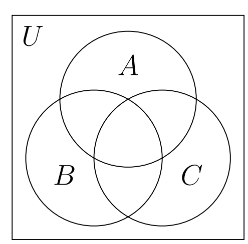
iv.) $A \cap B \cap C$



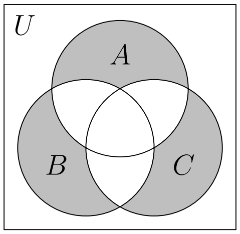
vi.) $(A \cap (A - C)) \cup B$

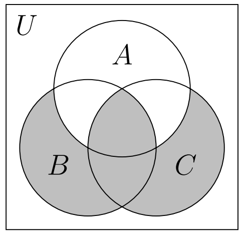


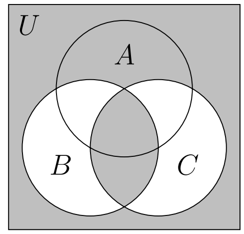
viii.) $A \cap (A - (C \cup B))$

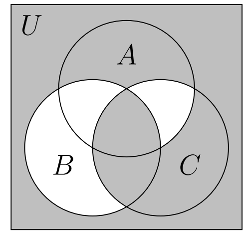


Exercise 4. Give an expression that describes each shaded region.

i.)


ii.)


iii.)


iv.)


i.)

ii.)

iii.)

iv.)

Exercise 5. Simplify each expression (if possible), then express the set in set builder notation.

i.) $A \cup A \cup B$

ii.) $B - (B - A)$

iii.) $(B - B) - A$

iv.) $B - (C - A)$

v.) $C - \overline{A}$

vi.) $\overline{A} \cap B \cap C$

vii.) $A \cup (B \cap \overline{B})$

viii.) $\overline{A \cup B}$

Homework. Due Monday, February 1 at 6pm.

- Respond to the poll: `https://www.surveymonkey.com/r/GassertStudentVideoConsent`
- Read Sections 1.5–7 from the text.
- Complete the following exercises (add these to your Overleaf file with the other book problems).
  - Section 1.5: 4 (all).
  - Section 1.6: 2 (all).
  - Section 1.7: 7, 8. (You do not have to turn in the pictures.)

3.5 Indexed sets

Section 1.8 of Hammack. Namely, intersections and unions of collections of sets (as opposed to pairs of sets).

Exercise 2 is a bit ambiguous, especially part (i).

MATH 2001
INDEXED SETS

Homework. Due Friday, February 5 at 6pm.

- Respond to the poll: `https://www.surveymonkey.com/r/GassertStudentVideoConsent` if you haven't done so already
- Read Sections 1.8 from the text.
- Complete the following exercises (add these to your Overleaf file with the other book problems).
  - Section 1.8: 2, 3, 4, 6, 9.

Exercise 1. Suppose $A_1 = \{a, b, c\}$, $A_2 = \{c, d, f\}$, and $A_3 = \{b, c, e\}$. Then

i. $\bigcup_{i=1}^{3} A_i =$

ii. $\bigcap_{i=1}^{3} A_i =$

Exercise 2. Sketch each of the following sets (in $\mathbb{R}$ or in $\mathbb{R}^2$).

i. $\bigcup_{n\in\mathbb{N}} \{(n, n)\}$

ii. $\bigcup_{n\in\mathbb{N}} \{n, -n\}$

iii. $\bigcup_{x=3}^{5} ([1, x] \times \mathbb{R})$

iv. $\bigcap_{n=2}^{4} [n^{-1}, n]$

v. $\bigcap_{n=1}^{\infty} (-n, n]$.

Exercise 3.

Let A be a set, and consider the following sets derived from A:

$$X_1 = \{x : x \subseteq A\} \qquad X_2 = \bigcup_{x \in X_1} x \qquad X_3 = \bigcup_{x \in X_1} \{x\}$$

$$X_4 = \{x : x \in A\} \qquad X_5 = \bigcup_{x \in X_4} x \qquad X_6 = \bigcup_{x \in X_4} \{x\}.$$

(1) Translate the definitions of X_1 and X_2 into English phrases. (How would you read each statement out loud? "X_1 is ...")

(2) Write out the sets $X_1, X_2, \ldots, X_6$ in the case where $A = \{\mathbb{N}, \mathbb{Z}\}$.

(3) Given an arbitrary set A, one of the X_i has a nonsensical definition. Which set is it, and why does its definition not make sense?

(4) Given a set A for which all of the X_i are defined, which statement *best* describes each set? (Write X_1 next to the statement that best describes it, etc.)

(a) X_i is a subset of A.

(b) X_i is equal to A.

(c) A is a subset of X_i.

(d) None of the above.

3.6 First proof

Up until now, I have just been following Chapter 1 of Hammack, but now I am jumping up to Chapter 8. We just learned all this set vocabulary, and since this is a proof course, I feel that it is time to start writing proofs.

This worksheet accompanies a set of slides. The task is simple: prove that if A and B are sets and $A = B$, then $A \subseteq B$.

The idea is to get the students to lay out the relevant definitions in front of them, and arrange the statements in order as you would with a series of dominos. (Take a look at the slides if this is unclear.) By laying out and arranging the definitions, we have created a framework for the proof. Once we have the framework in place, we can write a formal proof.

The source file for the slides is `2001-6a-FirstProof.tex`.

This is also the point in the course where I started the Proof Portfolio.

MATH 2001
FIRST PROOF

Homework. Due Monday, February 8 at 6pm.

- Begin a new Overleaf document using the Proof Portfolio Template found on the course website.
  - Course website: `http://math.colorado.edu/~thga2182/Discrete_math/16S/`
  - Under the Resources heading, click on the `Overleaf` link next to 'Proof portfolio template'.
  - Copy the template for the Proof Portfolio into a new file, and send me the Read & Edit link to your file.

Exercise 1. Write a new definition of *set equality* that involves explicit statements regarding the elements in the sets.

Definition.

Exercise 2. Suppose A and B are sets, and $A = B$. What can you say specifically about the elements in A and B?

Exercise 3. Suppose A and B are sets. What would you have to do to prove that $A = B$?

Exercise 4. Write a new definition of *subset* that involves explicit statements regarding the elements in the sets.

Definition.

Exercise 5. Suppose A and B are sets, and $A \subseteq B$. What can you say specifically about the elements in A and B?

Exercise 6. Suppose A and B are sets. What would you have to do to prove that $A \subseteq B$?

Exercise 7. Prove the following theorem.

Theorem. *If A and B are sets, and $A = B$, then $A \subseteq B$.*

Proof.

$\square$

Hello world!

A First Proof

Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

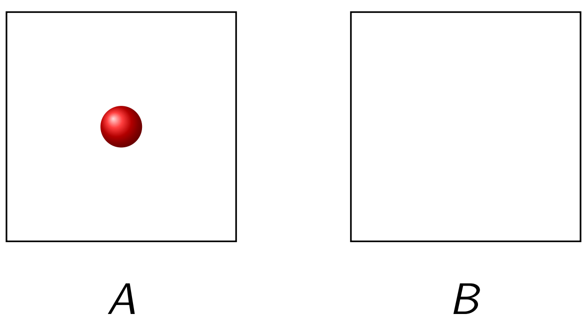
$$\text{sets are equal} \qquad \Leftrightarrow \qquad \text{elements are the same}$$

Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

$$\text{sets are equal} \qquad \Leftrightarrow \qquad \text{elements are the same}$$

If $A = B$ and $x \in A$, then ...



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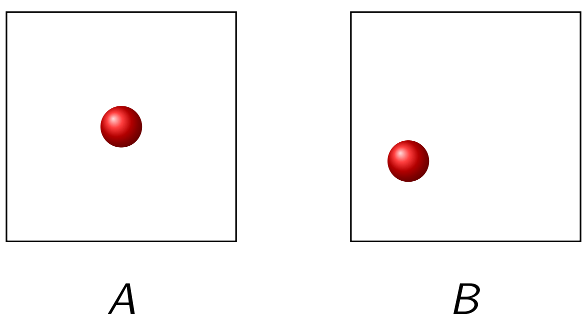
If $A = B$ and $x \in A$, then $x \in B$.

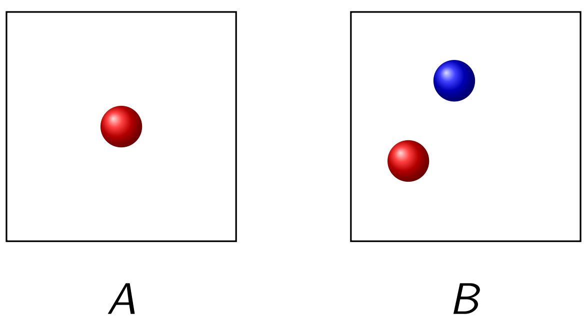


Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

sets are equal $\Leftrightarrow$ elements are the same

If $A = B$ and $x \in A$, then $x \in B$.
If $A = B$ and $x \in B$, then ...



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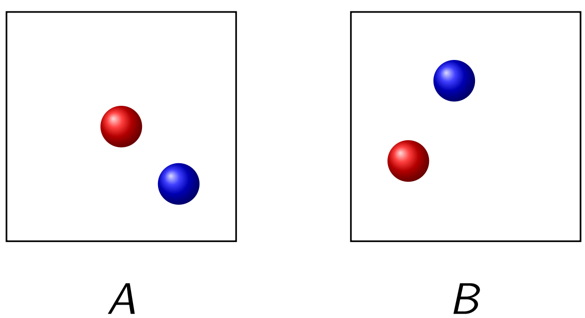


Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

Know: Given that $A = B$, we *know* that

- if $x \in A$, then $x \in B$, and
- if $x \in B$, then $x \in A$.

Definition of set equality

$$\begin{tikzcd}
 & x \in A \Rightarrow x \in B \\
A = B \arrow[ur, "\text{know}"] \arrow[dr, "\text{know}"'] & \\
 & x \in B \Rightarrow x \in A
\end{tikzcd}$$

Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

Know: Given that $A = B$, we *know* that

- if $x \in A$, then $x \in B$, and
- if $x \in B$, then $x \in A$.

Show: How would you show that two sets are equal?

Definition (*Set equality*)

Let A and B be sets, then $A = B$ if A and B contain exactly the same elements.

Know: Given that $A = B$, we *know* that

- if $x \in A$, then $x \in B$, and
- if $x \in B$, then $x \in A$.

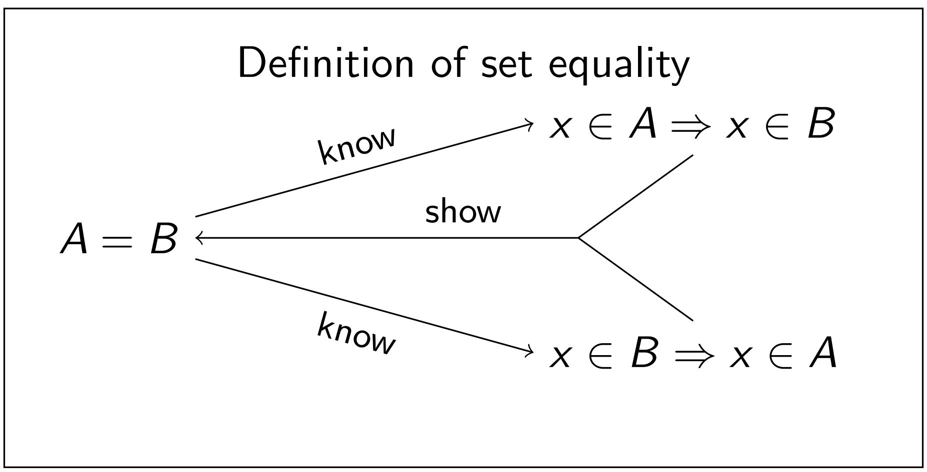
Show: How would you show that two sets are equal?
If we can *show* the following:

- if $x \in A$, then $x \in B$, and
- if $x \in B$, then $x \in A$,

then we would know that $A = B$.

Definition

Let A and B be sets, then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.



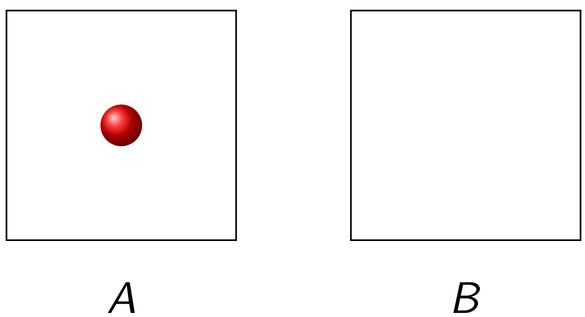
Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if every element of A is also an element of B.

Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if every element of A is also an element of B.

If $A \subseteq B$ and $x \in A$, then ...



Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if every element of A is also an element of B.

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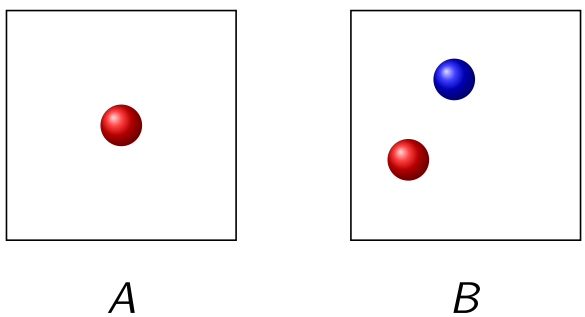
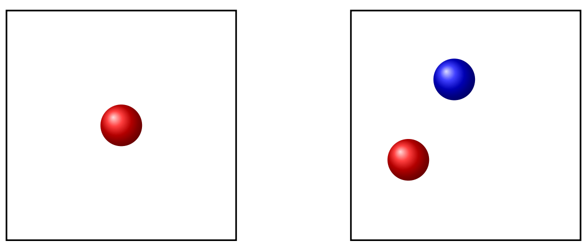


Know: Given that $A \subseteq B$, we *know* that if $x \in A$, then $x \in B$.

Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if every element of A is also an element of B.

If $A \subseteq B$ and $x \in A$, then $x \in B$.



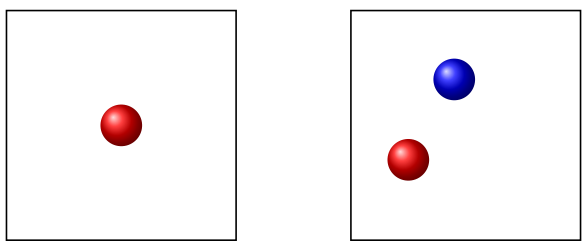
A B

Know: Given that $A \subseteq B$, we *know* that if $x \in A$, then $x \in B$.
Show: Knowing ...
... would *show* that $A \subseteq B$.

Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if every element of A is also an element of B.

If $A \subseteq B$ and $x \in A$, then $x \in B$.



A $\quad\quad$ B

Know: Given that $A \subseteq B$, we *know* that if $x \in A$, then $x \in B$.
Show: Knowing if $x \in A$, then $x \in B$
would *show* that $A \subseteq B$.

Definition (*Subset*)

Let A and B be sets, then $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

Definition of subset

$$\begin{tikzcd} A \subseteq B \arrow[r, bend left, "\text{know}"] & x \in A \Rightarrow x \in B \arrow[l, bend left, "\text{show}"] \end{tikzcd}$$

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

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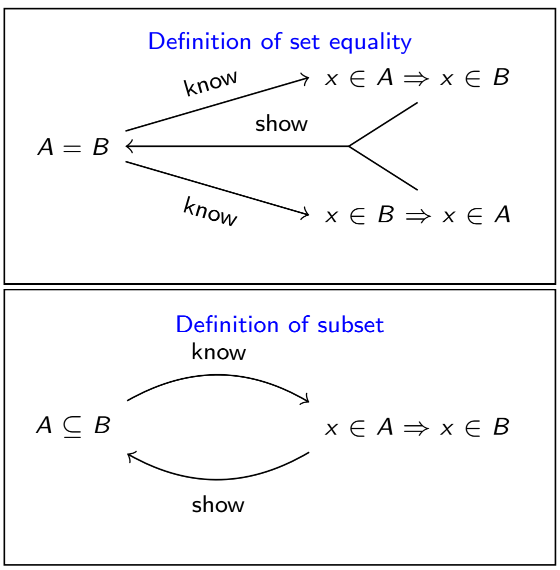

Prove the following.

Theorem

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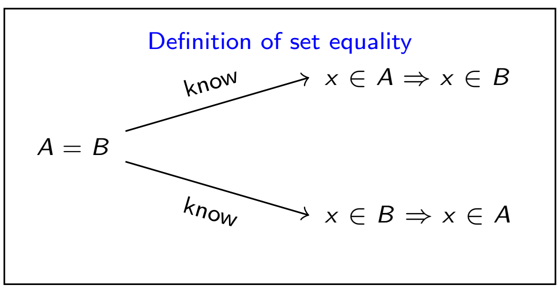
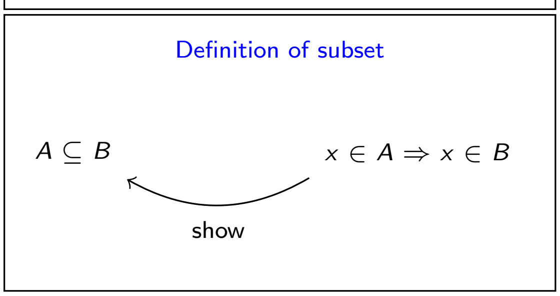




Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."

Definition of set equality

$$\begin{tikzcd} & x \in A \Rightarrow x \in B \quad \star \\ A = B \arrow[ur, "\text{know}"] \arrow[dr, "\text{know}"'] & \\ & x \in B \Rightarrow x \in A \quad \star \end{tikzcd}$$

Definition of subset

$$\star \quad x \in A \Rightarrow x \in B \xrightarrow[\text{show}]{} A \subseteq B$$

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."

Definition of set equality

$$\begin{tikzcd}
 & x \in A \Rightarrow x \in B \quad \star \\
A = B \arrow[ur, "\text{know}"] \arrow[dr, "\text{know}"'] & \\
 & x \in B \Rightarrow x \in A \quad \star
\end{tikzcd}$$

Definition of subset

$$\star \quad x \in A \Rightarrow x \in B \xrightarrow[\text{show}]{} A \subseteq B$$

Proof.

Suppose that A and B are sets, and $A = B$.

□

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."

Definition of set equality

$$\begin{tikzcd} & x \in A \Rightarrow x \in B \quad \star \\ A = B \arrow[ur, "\text{know}"] \arrow[dr, "\text{know}"'] & \\ & x \in B \Rightarrow x \in A \quad \star \end{tikzcd}$$

Definition of subset

$$\star \quad x \in A \Rightarrow x \in B \xrightarrow[\text{show}]{} A \subseteq B$$

Proof.

Suppose that A and B are sets, and $A = B$. We prove that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

$\square$

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."

Definition of set equality

$$\begin{tikzcd} & x \in A \Rightarrow x \in B \quad \star \\ A = B \arrow[ur, "\text{know}"] \arrow[dr, "\text{know}"'] & \\ & x \in B \Rightarrow x \in A \quad \star \end{tikzcd}$$

Definition of subset

$$\star \quad x \in A \Rightarrow x \in B \xrightarrow[\text{show}]{} A \subseteq B$$

Proof.

Suppose that A and B are sets, and $A = B$. We prove that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

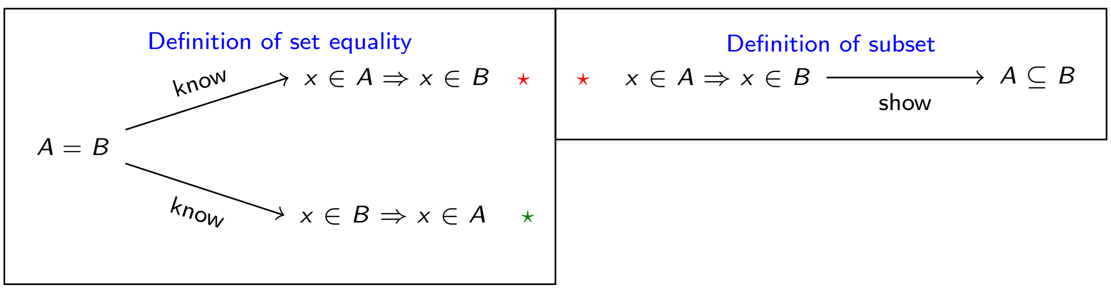
Suppose $x \in A$. $\square$

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."



Proof.

Suppose that A and B are sets, and $A = B$. We prove that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

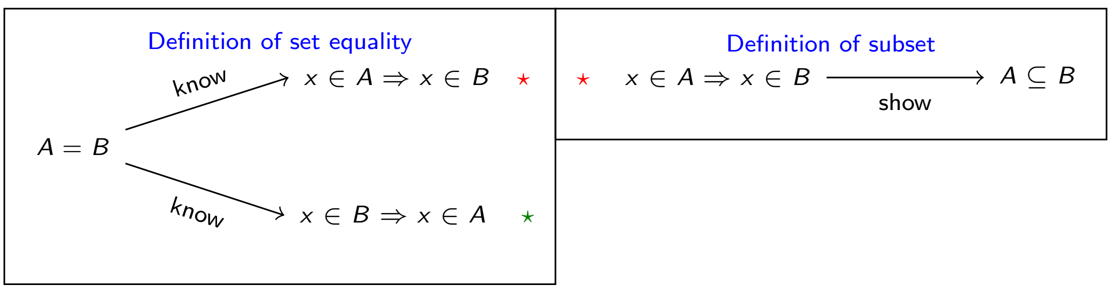
Suppose $x \in A$. By the definition of set equality, if $x \in A$, then $x \in B$. $\square$

Prove the following.

Theorem

Let A and B be sets. If $A = B$, then $A \subseteq B$.

"*Knowing* that A and B are sets and $A = B$, *show* that $A \subseteq B$."



Proof.

Suppose that A and B are sets, and $A = B$. We prove that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

Suppose $x \in A$. By the definition of set equality, if $x \in A$, then $x \in B$. Thus by the definition of subset, we conclude that $A \subseteq B$. □

3.7 Second proof

The goal of this worksheet is to prove the statement: if A and B are sets, then $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$. The goal was to have students, in the style of the previous lecture, to have students make "domino" cards with the definitions, then arrange them to piece together this argument.

Turned out, this was moving too fast for the students. Most of them did not have a strong enough grasp of how to use and apply the definitions appropriately for building arguments. So the next two worksheets address those concerns.

If done again, I would move this worksheet down a few.

MATH 2001
PROOFS

Homework. Due Friday, February 12 at 6pm.

- Complete Proof 2 (originally due today, but pushed back to Friday).
- Revise Proof 1.
- Add Exercise 1 on this worksheet in your Book Problems Overleaf file.
- Read sections 8.1 and 8.2 and definition 4.4. (Skip examples 8.3 and 8.7).

Definition (Subset)**.**

Definition (Union)**.**

Theorem. *If A and B are sets, then $A \subseteq A \cup B$.*

Proof.

$\square$

Definition (Power set)**.**

Theorem. *If A and B are sets, then $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.*

Definition (Divides)**.** Suppose $a, b \in \mathbb{Z}$, then a *divides* b if $ac = b$ for some $c \in \mathbb{Z}$.

Notation. We write $a \mid b$ (`a \mid b`) to denote that a divides b or that a is a *divisor* of b.

Exercise 1. Demonstrate why the statement is true or explain why the statement is false.

T	**F**	:	$4 \mid 20$	**T**	**F**	:	$0 \mid 0$
T	**F**	:	$3 \mid 11$	**T**	**F**	:	$\frac{1}{2} \mid 10$
T	**F**	:	$0 \mid 33$	**T**	**F**	:	$\frac{1}{3} \mid \frac{2}{3}$

Exercise 2. Prove that $\{x \in \mathbb{Z} : 55 \mid x\} \subseteq \{x \in \mathbb{Z} : 11 \mid x\}$.

3.8 Definition review

In this worksheet, the first focus is to break down each set definition into statements about elements. The goal is that once we have each statement, we can link them up to build an argument. The exercises involve building the skeleton of a proof using the statements coming directly from the definitions.

MATH 2001
DEFINITIONS: REVIEW

Definition 1 (Set equality)**.** If A and B are sets, then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

Definition 2 (Subset)**.** If A and B are sets, then $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

Definition 3 (Power set)**.** If A is a set, then $\mathscr{P}(A) = \{x : x \subseteq A\}$.

Definition 4 (Union)**.** If A and B are sets, then $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Definition 5 (Finite and infinite unions)**.** If $\{A_i\}$ is a collection of sets indexed by I, then

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}.$$

Definition 6 (Intersection)**.** If A and B are sets, then $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Definition 7 (Finite and infinite intersection)**.** If $\{A_i\}$ is a collection of sets indexed by I, then

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}.$$

Definition 8 (Set difference)**.** If A and B are sets, then $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Definition 9 (Complement)**.** If A is a set, then $\overline{A} = \{x : x \notin A\}$.

Exercise 1. Fill out the right side of each block.

Set equality

$A = B \quad \Leftrightarrow$

Subset

$A \subseteq B \quad \Leftrightarrow$

Power set

$x \in \mathscr{P}(A) \quad \Leftrightarrow$

Union

$x \in A \cup B \quad \Leftrightarrow$

In/finite union

$x \in \bigcup_{i \in I} A_i \quad \Leftrightarrow$

Intersection

$x \in A \cap B \quad \Leftrightarrow$

In/finite intersection

$x \in \bigcap_{i \in I} A_i \quad \Leftrightarrow$

Set difference

$x \in A - B \quad \Leftrightarrow$

Complement

$x \in \overline{A} \quad \Leftrightarrow$

All of the proof that we have been writing recently have been of the form, “prove that $A \subseteq B$.” In order to prove that $A \subseteq B$, we show that A and B satisfy the definition of subset. Namely, we show that if $x \in A$, then $x \in B$. In particular, the body of the proof should start with the statement, “suppose $x \in A$” (or “if $x \in A$”). Then after a series of logical deductions, the proof ends once we conclude that $x \in B$.

Exercise 2. Arrange the following statements to give an outline for a proof that $(A \cap B) - C \subseteq (A - C) \cap (B - C)$. Justify each statement by citing the appropriate definition.

_____ $\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)

a. $x \in A$ and $x \in B$ and $x \notin C$
b. $x \in A \cap B$ and $x \notin C$
c. $x \in (A \cap B) - C$
d. $x \in A - C$ and $x \in B - C$
e. $x \in (A - C) \cap (B - C)$

Exercise 3. Since unions are ‘or’ statements, proofs involving unions often break into multiple cases. Arrange the statements to prove that $(A - C) \cup (B - C) \subseteq (A \cup B) - C$. Justify each statement by citing the appropriate definition.

_____ $\Rightarrow$ _____ (by definition of)
Case 1: _____ $\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
Case 2: _____ $\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)
$\Rightarrow$ _____ (by definition of)

a. $x \in A - C$
b. $x \in B - C$
c. $x \in A \cup B$ and $x \notin C$
d. $x \in (A \cup B) - C$
e. $x \in B$ and $x \notin C$
f. $x \in A$ and $x \notin C$
g. $x \in A$ or $x \in B$, and $x \notin C$
h. $x \in A - C$ or $x \in B - C$
i. $x \in (A - C) \cup (B - C)$

Exercise 4. In a similar fashion, sketch proofs for the following statements. In some cases, the justification for a step might not be a definition, but information that was given in the statement of the problem.

a. Prove that if $X \subseteq A \cap B$, then $X \subseteq A$ and $X \subseteq B$.
b. Prove that $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.
c. Prove that if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.

Upcoming deadlines:

- Due Friday, Feb 19: first draft of Proof 4.
- Due Monday, Feb 22: final draft of Proof 1 and second draft of proof 2.
- Due Wednesday, Feb 24: second draft of proof 3.

3.9 Definition review II

I forgot one definition, so one more round of this type of worksheet. Turns out it was a good one to forget. The definition I forgot was Cartesian product, so now we have exercises where the generic element of a set is an ordered pair.

MATH 2001
DEFINITIONS: REVIEW II

Definition 1 (Cartesian product)**.** If A and B are a sets, then $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Cartesian product	
$(a, b) \in A \times B$	$\Leftrightarrow$

Exercise 1. Arrange the following statements to give an outline for a proof that $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$. Justify each statement by citing the appropriate definition.

Proof. (Start by a short introduction defining the variables and describing what will be proved.)

_____ $\Rightarrow$ _____	(by definition of)	a. $x \in A$, $x \in B$, and $y \in C$.
$\Rightarrow$ _____	(by definition of)	b. $(x, y) \in (A \cap B) \times C$
$\Rightarrow$ _____	(by definition of)	c. $(x, y) \in A \times C$ and $(x, y) \in B \times C$
$\Rightarrow$ _____	(by definition of)	d. $x \in A \cap B$ and $y \in C$
		e. $(x, y) \in (A \times C) \cap (B \times C)$

$\square$

Exercise 2. In a similar fashion, sketch proof for the statement $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$. Include a brief introduction for each proof.

In Proof 2 in the Proof Portfolio, I stated the following theorem.

Theorem. *If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

This theorem (showing that two sets are subsets of each other) is the most common technique for proving that two sets are equal.

Exercise 3. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by proving that

a. $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$, and

b. $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Upcoming deadlines:

- Due Friday, Feb 19: first draft of Proof 4.
- Due Monday, Feb 22: final draft of Proof 1 and second draft of proof 2.
- Due Wednesday, Feb 24: second draft of proof 3, and first draft of proof 5.
- Due Friday, Feb 26: second draft of proof 4.

3.-1 Second proof II

Here is another set of slides which I did not use in the course this semester, but it does fit somewhere into this section of worksheets. This is another presentation of how we can use the definitions as dominos to build a proof.

I wrote this lecture for my last last fall (2015), and when I gave it, I also printed out a pile of black definition cards for the students to fill out during the lecture so that they could physically move the cards around to build the proofs. (See `2001Flashcards2.tex` in the `Flachcard` folder.) The work to format, print, and cut these cards too much for me to continue to do beyond this lecture. I had hoped to inspire students to continue using this idea throughout the whole semester

The source file for these slides is `2001-7b-SecondProof.tex`.

Prove the following.

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

That is, we are proving that $A = \overline{\overline{A}}$.

Prove the following.

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

That is, we are proving that $A = \overline{\overline{A}}$.
Why is that?

$$C = \overline{B} = \overline{\overline{A}}$$

(replace B with $\overline{A}$ in the last equality)

Prove the following.

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

That is, we are proving that $A = \overline{\overline{A}}$.
Why is that?

$$C = \overline{B} = \overline{\overline{A}}$$

(replace B with $\overline{A}$ in the last equality)
Proving $A = C$ is the same as showing $A = \overline{\overline{A}}$.

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set equality)

Let A and B be sets. Then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

Given $A = B$, we know

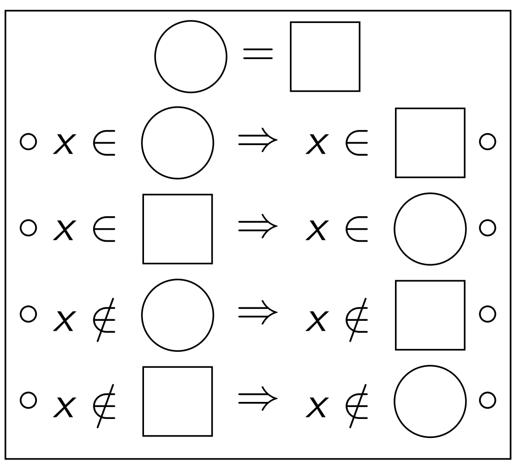
$$\begin{array}{ccccc} \circ & x \in A & \Rightarrow & x \in B & \circ \\ \circ & x \in B & \Rightarrow & x \in A & \circ \\ \circ & x \notin A & \Rightarrow & x \notin B & \circ \\ \circ & x \notin B & \Rightarrow & x \notin A & \circ \end{array}$$

Theorem

Let $A, B,$ and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set equality)

Let A and B be sets. Then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.



Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set equality)

Let A and B be sets. Then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

Given $B = \overline{A}$, we know

∘	$x \in B$	$\Rightarrow$	$x \in \overline{A}$	∘
∘	$x \in \overline{A}$	$\Rightarrow$	$x \in B$	∘
∘	$x \notin B$	$\Rightarrow$	$x \notin \overline{A}$	∘
∘	$x \notin \overline{A}$	$\Rightarrow$	$x \notin B$	∘

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set equality)

Let A and B be sets. Then $A = B$ if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

Given $C = \overline{B}$, we know

◦	$x \in C$	$\Rightarrow$	$x \in \overline{B}$	◦
◦	$x \in \overline{B}$	$\Rightarrow$	$x \in C$	◦
◦	$x \notin C$	$\Rightarrow$	$x \notin \overline{B}$	◦
◦	$x \notin \overline{B}$	$\Rightarrow$	$x \notin C$	◦

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set complement)

Let A be a set. Then $\overline{A} = \{x : x \notin A\}$.

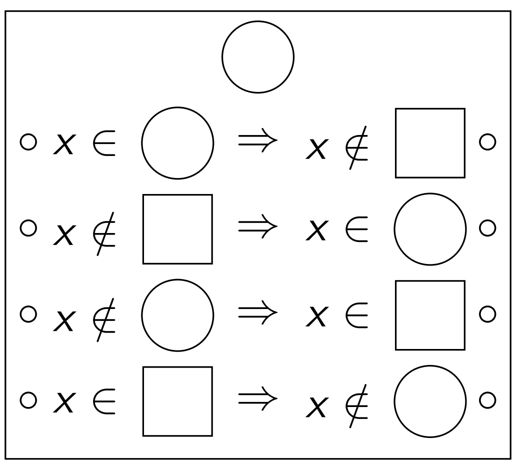
Given a set A, we know				
◦	$x \in A$	$\Rightarrow$	$x \notin \overline{A}$	◦
◦	$x \notin \overline{A}$	$\Rightarrow$	$x \in A$	◦
◦	$x \notin A$	$\Rightarrow$	$x \in \overline{A}$	◦
◦	$x \in \overline{A}$	$\Rightarrow$	$x \notin A$	◦

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set complement)

Let A be a set. Then $\overline{A} = \{x : x \notin A\}$.



Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set complement)

Let A be a set. Then $\overline{A} = \{x : x \notin A\}$.

Given the set A, we know

- $x \in A \Rightarrow x \notin \overline{A}$
- $x \notin \overline{A} \Rightarrow x \in A$
- $x \notin A \Rightarrow x \in \overline{A}$
- $x \in \overline{A} \Rightarrow x \notin A$

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Definition (Set complement)

Let A be a set. Then $\overline{A} = \{x : x \notin A\}$.

Given the set B, we know

- $x \in B \Rightarrow x \notin \overline{B}$
- $x \notin \overline{B} \Rightarrow x \in B$
- $x \notin B \Rightarrow x \in \overline{B}$
- $x \in \overline{B} \Rightarrow x \notin B$

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

$\ldots$

$\square$

Theorem (Double inclusion principle)

Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Theorem

Let A, B, and C be sets where $B = \overline{A}$ and $C = \overline{B}$. Then $A = C$.

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

($C \subseteq A$): In order to show that $C \subseteq A$, we show that every element of C is an element of A. Namely, $x \in C \Rightarrow x \in A$.

$\ldots$ $\square$

Definition (Subset)

Let A and B be sets. Then $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

Given $B = \overline{A}$, we know		
$x \in B$	$\Rightarrow$	$x \in \overline{A}$
$x \in \overline{A}$	$\Rightarrow$	$x \in B$
$x \notin B$	$\Rightarrow$	$x \notin \overline{A}$
$x \notin \overline{A}$	$\Rightarrow$	$x \notin B$

Given $C = \overline{B}$, we know		
$x \in C$	$\Rightarrow$	$x \in \overline{B}$
$x \in \overline{B}$	$\Rightarrow$	$x \in C$
$x \notin C$	$\Rightarrow$	$x \notin \overline{B}$
$x \notin \overline{B}$	$\Rightarrow$	$x \notin C$

Given the set A, we know		
$x \in A$	$\Rightarrow$	$x \notin \overline{A}$
$x \notin \overline{A}$	$\Rightarrow$	$x \in A$
$x \notin A$	$\Rightarrow$	$x \in \overline{A}$
$x \in \overline{A}$	$\Rightarrow$	$x \notin A$

Given the set B, we know		
$x \in B$	$\Rightarrow$	$x \notin \overline{B}$
$x \notin \overline{B}$	$\Rightarrow$	$x \in B$
$x \notin B$	$\Rightarrow$	$x \in \overline{B}$
$x \in \overline{B}$	$\Rightarrow$	$x \notin B$

$$x \in C \Rightarrow$$

$$\vdots$$

$$\Rightarrow x \in A$$

Given $B = \overline{A}$, we know	Given $C = \overline{B}$, we know	Given the set A, we know	Given the set B, we know
∘ $x \in B \Rightarrow x \in \overline{A}$ ∘	$\boxed{∘\ x \in C \Rightarrow x \in \overline{B}\ ∘}$	∘ $x \in A \Rightarrow x \notin \overline{A}$ ∘	∘ $x \in B \Rightarrow x \notin \overline{B}$ ∘
∘ $x \in \overline{A} \Rightarrow x \in B$ ∘	∘ $x \in \overline{B} \Rightarrow x \in C$ ∘	∘ $x \notin \overline{A} \Rightarrow x \in A$ ∘	∘ $x \notin \overline{B} \Rightarrow x \in B$ ∘
∘ $x \notin B \Rightarrow x \notin \overline{A}$ ∘	∘ $x \notin C \Rightarrow x \notin \overline{B}$ ∘	∘ $x \notin A \Rightarrow x \in \overline{A}$ ∘	∘ $x \notin B \Rightarrow x \in \overline{B}$ ∘
∘ $x \notin \overline{A} \Rightarrow x \notin B$ ∘	∘ $x \notin \overline{B} \Rightarrow x \notin C$ ∘	∘ $x \in \overline{A} \Rightarrow x \notin A$ ∘	∘ $x \in \overline{B} \Rightarrow x \notin B$ ∘

$$x \in C \Rightarrow x \in \overline{B} \qquad \text{(since } C = \overline{B}\text{)}$$

$$\vdots$$

$$\Rightarrow x \in A$$

Given $B = \overline{A}$, we know
∘ $x \in B \quad \Rightarrow \quad x \in \overline{A}$ ∘
∘ $x \in \overline{A} \quad \Rightarrow \quad x \in B$ ∘
∘ $x \notin B \quad \Rightarrow \quad x \notin \overline{A}$ ∘
∘ $x \notin \overline{A} \quad \Rightarrow \quad x \notin B$ ∘

Given $C = \overline{B}$, we know
∘ $\boxed{x \in C \quad \Rightarrow \quad x \in \overline{B}}$ ∘
∘ $x \in \overline{B} \quad \Rightarrow \quad x \in C$ ∘
∘ $x \notin C \quad \Rightarrow \quad x \notin \overline{B}$ ∘
∘ $x \notin \overline{B} \quad \Rightarrow \quad x \notin C$ ∘

Given the set A, we know
∘ $x \in A \quad \Rightarrow \quad x \notin \overline{A}$ ∘
∘ $x \notin \overline{A} \quad \Rightarrow \quad x \in A$ ∘
∘ $x \notin A \quad \Rightarrow \quad x \in \overline{A}$ ∘
∘ $x \in \overline{A} \quad \Rightarrow \quad x \notin A$ ∘

Given the set B, we know
∘ $x \in B \quad \Rightarrow \quad x \notin \overline{B}$ ∘
∘ $x \notin \overline{B} \quad \Rightarrow \quad x \in B$ ∘
∘ $x \notin B \quad \Rightarrow \quad x \in \overline{B}$ ∘
∘ $\boxed{x \in \overline{B} \quad \Rightarrow \quad x \notin B}$ ∘

$$
\begin{aligned}
x \in C &\Rightarrow x \in \overline{B} && \text{(since } C = \overline{B}\text{)} \\
&\Rightarrow x \notin B && \text{(by def. of converse of } B\text{)} \\
&\vdots \\
&\Rightarrow x \in A
\end{aligned}
$$

Given $B = \overline{A}$, we know		
$x \in B$	$\Rightarrow$	$x \in \overline{A}$
$x \in \overline{A}$	$\Rightarrow$	$x \in B$
$\boxed{x \notin B}$	$\boxed{\Rightarrow}$	$\boxed{x \notin \overline{A}}$
$x \notin \overline{A}$	$\Rightarrow$	$x \notin B$

Given $C = \overline{B}$, we know		
$\boxed{x \in C}$	$\boxed{\Rightarrow}$	$\boxed{x \in \overline{B}}$
$x \in \overline{B}$	$\Rightarrow$	$x \in C$
$x \notin C$	$\Rightarrow$	$x \notin \overline{B}$
$x \notin \overline{B}$	$\Rightarrow$	$x \notin C$

Given the set A, we know		
$x \in A$	$\Rightarrow$	$x \notin \overline{A}$
$x \notin \overline{A}$	$\Rightarrow$	$x \in A$
$x \notin A$	$\Rightarrow$	$x \in \overline{A}$
$x \in \overline{A}$	$\Rightarrow$	$x \notin A$

Given the set B, we know		
$x \in B$	$\Rightarrow$	$x \notin \overline{B}$
$x \notin \overline{B}$	$\Rightarrow$	$x \in B$
$x \notin B$	$\Rightarrow$	$x \in \overline{B}$
$\boxed{x \in \overline{B}}$	$\boxed{\Rightarrow}$	$\boxed{x \notin B}$

$$
\begin{aligned}
x \in C &\Rightarrow x \in \overline{B} && \text{(since } C = \overline{B}\text{)}\\
&\Rightarrow x \notin B && \text{(by def. of converse of } B\text{)}\\
&\Rightarrow x \notin \overline{A} && \text{(since } B = \overline{A}\text{)}\\
&\vdots\\
&\Rightarrow x \in A
\end{aligned}
$$

Given $B = \overline{A}$, we know

- $x \in B \Rightarrow x \in \overline{A}$
- $x \in \overline{A} \Rightarrow x \in B$
- $x \notin B \Rightarrow x \notin \overline{A}$
- $x \notin \overline{A} \Rightarrow x \notin B$

Given $C = \overline{B}$, we know

- $x \in C \Rightarrow x \in \overline{B}$
- $x \in \overline{B} \Rightarrow x \in C$
- $x \notin C \Rightarrow x \notin \overline{B}$
- $x \notin \overline{B} \Rightarrow x \notin C$

Given the set A, we know

- $x \in A \Rightarrow x \notin \overline{A}$
- $x \notin \overline{A} \Rightarrow x \in A$
- $x \notin A \Rightarrow x \in \overline{A}$
- $x \in \overline{A} \Rightarrow x \notin A$

Given the set B, we know

- $x \in B \Rightarrow x \notin \overline{B}$
- $x \notin \overline{B} \Rightarrow x \in B$
- $x \notin B \Rightarrow x \in \overline{B}$
- $x \in \overline{B} \Rightarrow x \notin B$

$$\begin{aligned}
x \in C \Rightarrow x &\in \overline{B} && \text{(since } C = \overline{B}\text{)} \\
\Rightarrow x &\notin B && \text{(by def. of converse of } B\text{)} \\
\Rightarrow x &\notin \overline{A} && \text{(since } B = \overline{A}\text{)} \\
\Rightarrow x &\in A && \text{(by def. of converse of } A\text{)}
\end{aligned}$$

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

$(C \subseteq A)$: In order to show that $C \subseteq A$, we show that every element of C is an element of A. Namely, $x \in C \Rightarrow x \in A$.

Suppose $x \in C$.

$\square$

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

$(C \subseteq A)$: In order to show that $C \subseteq A$, we show that every element of C is an element of A. Namely, $x \in C \Rightarrow x \in A$.

Suppose $x \in C$. Then

$$\begin{aligned} x \in C \Rightarrow x \in \overline{B} & \qquad \text{(since } C = \overline{B}\text{)} \\ \Rightarrow x \notin B & \qquad \text{(by def. of converse of } B\text{)} \\ \Rightarrow x \notin \overline{A} & \qquad \text{(since } B = \overline{A}\text{)} \\ \Rightarrow x \in A & \qquad \text{(by def. of converse of } A\text{).} \end{aligned}$$

□

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

$(C \subseteq A)$: In order to show that $C \subseteq A$, we show that every element of C is an element of A. Namely, $x \in C \Rightarrow x \in A$.

Suppose $x \in C$. Then

$$\begin{aligned} x \in C &\Rightarrow x \in \overline{B} && (\text{since } C = \overline{B}) \\ &\Rightarrow x \notin B && (\text{by def. of converse of } B) \\ &\Rightarrow x \notin \overline{A} && (\text{since } B = \overline{A}) \\ &\Rightarrow x \in A && (\text{by def. of converse of } A). \end{aligned}$$

Thus $C \subseteq A$ by the definition of subset.

$\square$

Proof.

We use a double inclusion argument to prove that $A = C$. That is, we will show that $A \subseteq C$ and $C \subseteq A$.

$(C \subseteq A)$: In order to show that $C \subseteq A$, we show that every element of C is an element of A. Namely, $x \in C \Rightarrow x \in A$.

Suppose $x \in C$. Then

$$\begin{aligned}
x \in C &\Rightarrow x \in \overline{B} && \text{(since } C = \overline{B}\text{)} \\
&\Rightarrow x \notin B && \text{(by def. of converse of } B\text{)} \\
&\Rightarrow x \notin \overline{A} && \text{(since } B = \overline{A}\text{)} \\
&\Rightarrow x \in A && \text{(by def. of converse of } A\text{).}
\end{aligned}$$

Thus $C \subseteq A$ by the definition of subset.

$(A \subseteq C)$: [Exercise.]

Since $C \subseteq A$ and $A \subseteq C$, we conclude that $A = C$. $\square$

3.10 "For some"

This worksheet was written in response to correcting the language I was seeing in the students' proofs involving the "for some" statement. The definitions of even and odd are introduced in this worksheet as those definitions fit the theme.

"For some" is equivalent to "there exists." I have not discussed the existential quantifier (or any logical statement) yet, hence some of the confusion? Maybe I should cover logical statements.

MATH 2001
DEFINITIONS: "FOR SOME"

Definition (Divides)**.** If $a, b \in \mathbb{Z}$, then a *divides* b (denoted $a \mid b$) if $ac = b$ for some $c \in \mathbb{Z}$.

If $a \mid b$, then a is a *divisor* of b, and b is a *multiple* of a.

Definition (Even)**.** If $a \in \mathbb{Z}$, then a is *even* if $a = 2c$ for some $c \in \mathbb{Z}$.

Definition (Odd)**.** If $a \in \mathbb{Z}$, then a is *odd* if $a = 2c + 1$ for some $c \in \mathbb{Z}$.

When we say "for some $c \in \mathbb{Z}$," we mean that

1. there exists at least one value that satisfies the condition, and
2. we let c represent one of those values.

Once the value is named, we can use c as if it is the actual thing, and not just some arbitrary integer.

For example, *someone* is in control of this classroom. Let us call that person ♩♫ *Mr. F* ♩♫. Now, we can refer to Mr. F (♩♫) as if he is an actual person, even if we do not know his identity.

"Rita claims Mr. F is her uncle!" not "Rita claims Mr. F is her uncle for some person Mr. F!"

Similarly, saying $ac = b$ for some $c \in \mathbb{Z}$ means that

"c is a solution to $ax = b$," not "c is a solution to $ax = b$ for some $c \in \mathbb{Z}$."

Exercise 1. Prove that the sum of two odd integers is even.

Exercise 2. Prove that $\{8^n : n \in \mathbb{Q}\} = \{2^n : n \in \mathbb{Q}\}$.

Exercise 3. Prove that the product of an even integer with any other integer is even.

Exercise 4. Prove that if a is odd, then $8 \mid (a^2 - 1)$.

Upcoming deadlines:

- Due Wednesday, Feb 24: second draft of proof 3, and first draft of proof 5.
- Due Friday, Feb 26: final draft of proof 2, second draft of proof 4.
- Due Monday, Feb 29: final draft of proof 3, second draft of proof 5, first draft of proof 6.

As the number of proofs are piling up, from proof 6 onwards, I will only be giving one round of comments before final copies are due.

3.11 Negation

So let us do that—cover logical statements. And/or/implies statements and their negations.

This and the next several worksheets more or less cover the contents of Chapter 2 of Hammack. I do not go into all the formalities; I just provide the rules and give examples.

MATH 2001
STATEMENTS AND NEGATION

Simple statements: "P". A statement (generally denoted P) is an expression that is decidedly true or false. By negating a statement ($\neg P$), we change its meaning from true to false, or false to true. A negation generally involves inserting or removing 'not' from the statement, though that is not always the case.

Exercise 1. Negate each of the following statements.

P	$\neg P$
I went to the store.	
No parking on week days.	
$\pi \in \mathbb{Z}$.	
$2+3>6$.	

And/or statements: "P and/or Q". A simply way to combine statements is to use the 'and' or 'or' conjunction. What happens when you negate such a statement?

Exercise 2. Negate each of the following statements.

P and/or Q	$\neg(P$ and/or $Q)$
I am 33 years old or I am 34 years old.	
3 is positive, but 4 is not.	
$\pi \in \mathbb{Q}$ and $\pi \notin \mathbb{Q}$.	
$2+3>6$ or $2+3<0$.	

Exercise 3. As a general rule:

- $\neg(P$ and $Q) =$
- $\neg(P$ or $Q) =$

(If you are familiar with the logical operators $\wedge$ and $\vee$, feel free to use them here.)

If-then statements: "if P, then Q". Perhaps the most common form of a statement in mathematics is the if-then statement. Since if-then statements are implications, the statement "if P, then Q" is equivalent to the statement "$P \Rightarrow Q$". The negation of an if-then statement is given by the following rule:

- $\neg(P \Rightarrow Q) = (P$ and $\neg Q)$.

Exercise 4. Negate each of the following statements. For each statement, indicate whether the statement is true or false.

$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
If it is Monday, then we have class.	
The light is green, so we can go.	
$x^2 \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$.	
If x^2 is odd, then x is odd.	

Converse. The *converse* of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Exercise 5. Write the converse of each of the following statements.

$P \Rightarrow Q$	$Q \Rightarrow P$
If it is Monday, then we have class.	
The light is green, so we can go.	
$x^2 \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$.	
If x^2 is odd, then x is odd.	

How is a statement related to its converse? Are they equivalent? Are they negations of each other? Or are they unrelated?

Contrapositive. The *contrapositive* of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

Exercise 6. Write the contrapositive of each of the following statements.

$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
If it is Monday, then we have class.	
The light is green, so we can go.	
$x^2 \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$.	
If x^2 is odd, then x is odd.	

How is a statement related to its contrapositive? Are they equivalent? Are they negations of each other? Or are they unrelated?

Upcoming deadlines:

- Due Monday, Feb 29: final draft of proof 3, second draft of proof 5, first draft of proof 6.
- Due Wednesday, Mar 2: final draft of proof 4, first draft of proof 7.
- Due Friday Mar 4: final draft of proof 5, final draft of proof 6.

As the number of proofs are piling up, from proof 6 onwards, I will only be giving one round of comments before final copies are due.

3.12 Proofs with negation

This is mainly proofs involving the complements of sets. Although the second half of the statement does involve a proof by contrapositive (which I have not officially discussed yet).

MATH 2001
STATEMENTS AND NEGATION

Exercise 1. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Exercise 2. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. (This is proof 7 in the proof portfolio.)

Exercise 3. Let $A_y = [-1-y, 1+y]$. Prove that $\bigcap_{\substack{y\geq 0,\\ y\in\mathbb{R}}} A_y = [-1, 1]$.

Exercise 4. Let $A_y = (-y, y) \subseteq \mathbb{R}$ (A_y is an open interval). What is $\bigcap_{y>0} A_y$? Prove your claim.

Upcoming deadlines:

- Due Friday, Mar 4: final draft of proof 4, first draft of proof 7.
- Due Monday Mar 7: final draft of proof 5, final draft of proof 6.

As the number of proofs are piling up, from proof 6 onwards, I will only be giving one round of comments before final copies are due.

3.13 Quantifiers

To round out the logic section, quantifiers and negations. A number of my examples involve the rearrangement or replacement of clauses. The intension it to stress that the wording of statements is necessarily precise, and that even a minor alteration can drastically change the meaning of the statement.

If used again, I would make each of the diagrams on the second page have the same number of circles and squares. It seems to be an unnecessary complication to figure out whether a statement becomes vacuous if the corresponding circle or square is missing.

MATH 2001
CONDITIONALS, QUANTIFIERS, AND NEGATION

As stated previously, if-then statements are very common. Due to the nuance and variation of the English language, there are many ways to restate an if-then statement. Depending on the context, some phrasings may be more appropriate than others. We have already seen one example: the statement "if P, then Q" is equivalent to the statement "$P \Rightarrow Q$". Here are a few more equivalent forms.

Exercise 1. Consider the following "if P, then Q" statement.

If $\underbrace{x = 1}_{P}$, then $\underbrace{x^2 = 1}_{Q}$.

Complete each of the following statements so that they are equivalent to the if-then statement above.

a. ______________________________ if ______________________________.

b. ______________________________ only if ______________________________.

c. ______________________________ implies ______________________________.

d. ______________________________ is implied by ______________________________.

e. ______________________________ whenever ______________________________.

f. ______________________________ is necessary when ______________________________.

g. ______________________________ is sufficient for ______________________________.

As long as we can identify the P and Q parts in statements like the ones above, we can follow the rules to form the negations, contrapositives, and converses of the statements as necessary.

Exercise 2. For the statement "$P \Rightarrow Q$", the contrapositive, negation, and converse are

- Contrapositive:
- Negation:
- Converse:

For each/all/every: "For all P, we have Q" or "Q for all P". Another common phrase is the "for all" statement. For example: "$x^2 < 0$ for all $x \in \mathbb{R}$." (Equivalently, "for all $x \in \mathbb{R}$, we have $x^2 < 0$.")

There exists: "There exists P such that Q". The "for all" statement implies that the statement is always true. On the other end of the spectrum, the "there exists" statement implies that the statement is true in at least one case (but may be false otherwise). For example: "there exists $c \in \mathbb{Z}$ such that $3c = 7$."

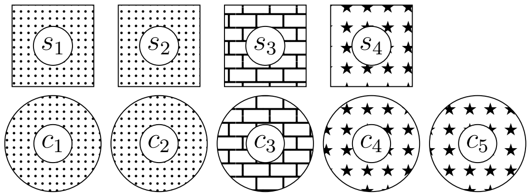
Of course, both my example are false statements. So by negating these statements, we will have true statements. How do we negate "for all" and "there exists"?

- $\neg$("for all P, we have Q") = "there exists P such that $\neg Q$"
- $\neg$("there exists P such that Q") = "for all P, we have $\neg Q$"

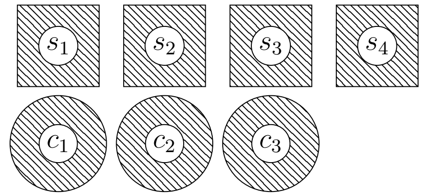
E.g. $\neg$("for all $x \in \mathbb{R}$, we have $x^2 < 0$") = "there exists $x \in \mathbb{R}$ such that $x^2 \geq 0$."

$\neg$("there exists $c \in \mathbb{Z}$ such that $3c = 7$") = "for all $c \in \mathbb{Z}$, we have $3c \neq 7$."

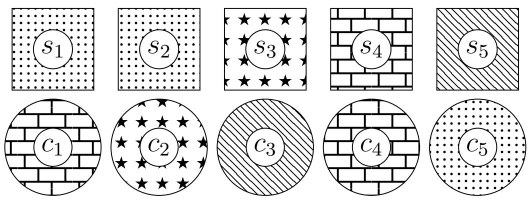
I.



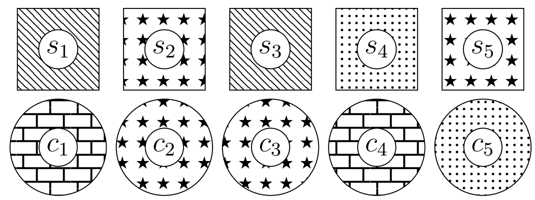
III.



II.



IV.



Let $P(s_i)$ and $P(c_i)$ denote the pattern on the i-th square or circle, respectively.

- For each of the following statements, write its contrapositive, negation, and converse where requested.
- Additional, identify which diagrams satisfy each statement.

Satisfied by

1. If $i = j$, then $P(s_i) = P(c_j)$.

   Contrapositive:

   Negation:

   Converse:

2. The pattern $P(c_i)$ is “lines” whenever $P(s_i)$ is “lines”.

   Contrapositive:

   Negation:

   Converse:

3. For each i, there exists a j for which $P(c_i) = P(s_j)$.

   Negation:

4. For each i, there exists a j for which $P(s_i) = P(c_j)$.

Negation:

5. There exists an i such that $P(s_j) = P(c_i)$ for all j.

Negation:

6. For all i and all j, we have $P(c_i) = P(s_j)$.

Negation:

7. There exists an i and there exists a j such that $P(c_i) = P(s_j)$.

Negation:

8. If there exists an i for which $P(c_i) = P(s_i)$, then there exists a $j \neq i$ such that $P(c_i) = P(c_j)$.

Contrapositive:

Negation:

Converse:

9. There exists an i for which $P(c_i)$ is "dots" only if there exists a j for which $P(s_j)$ is "dots".

Contrapositive:

Negation:

Converse:

Upcoming deadlines:

- Due Friday, Mar 4: final draft of proof 4, first draft of proof 7.
- Due Monday Mar 7: final draft of proof 5, final draft of proof 6.
- Due Wednesday Mar 9: final draft of proof 7, first draft of proof 8.

Exercise 3. Complete each of the following statements using the following clauses.

P: It rains heavily. Q: The streams flood their banks.

(You may need to alter how the clauses are phrased.)

a. If ____, then ____.

b. ____ if ____.

c. ____ only if ____.

d. ____ implies ____.

e. ____ implied by ____.

f. ____ whenever ____.

g. ____ necessary ____.

h. ____ sufficient ____.

3.14 Proofs by contradiction and contrapositive

Now formally introducing/recapping all proof styles.

Spacing on this worksheet is poor. I was in a rush making this one, so little thought went into it.

MATH 2001
CONTRAPOSITIVE AND CONTRADICTION

Upcoming deadlines:

- Due Wednesday, Mar 9: final draft of proof 7, first draft of proof 8.
- Due Monday, Mar 14: final draft of proof 8, first draft of proof 9.

Styles of proof.

- Direct:
- Contrapositive:
- Contradiction:

Prove the following.

1. There is no integer that is both even and odd.
2. If a^2 is even, then a is even.
3. The number $\sqrt{2}$ is irrational.
4. The number $\sqrt{3}$ is irrational.
5. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

3.15 Topology

Topology is not covered in Hammack.

I see this as a natural progression of the topics covered so far. We studied sets and logical statements. So now we combine logical operators and quantifiers to form more complex definitions about sets.

Admittedly, I teach this largely in part due to my experience teaching 3001 (analysis), which is a course that many 2001 students are directed into. These definitions are similar in complexity to those in 3001, and I want my students to have some experience working with these statements. A familiarity with open and closed sets does not hurt either.

There is a lot going on this worksheet, and I ended up not covering all of it. In the end, the only definitions that I care about are the basis of a topology, open sets, and closed sets. Initially, this was also going to serve as the motivation for proof by induction, as it is used to prove that a topology generated by a basis is indeed a topology. However, as I have mentioned already, I did not end up getting to this point.

If done again, I would drop all definitions other than basis, open, and closed. I would also cover partitions first, in part because every partition is a basis for a topology.

MATH 2001
TOPOLOGY

Upcoming deadlines:

- Due Monday, Mar 14: final draft of proof 8, first draft of proof 9.
- Due Wednesday, Mar 16: first draft of proof 10.
- Due Friday, Mar 18: final draft of proof 9.

Definition. Let A be a set, and let $\mathcal{T}$ be a set of subsets of A. The set $\mathcal{T}$ is a *topology on* A if $\mathcal{T}$ satisfies the following properties.

(1) The sets $\varnothing$ and A are elements of $\mathcal{T}$.

(2) The set $\mathcal{T}$ is closed under arbitrary union: if $S \subseteq \mathcal{T}$, then
$$\bigcup_{U \in S} U \in \mathcal{T}.$$
That is, the union of any number of elements in $\mathcal{T}$ (finite or infinite) is an element of $\mathcal{T}$.

(3) The set $\mathcal{T}$ is closed under finite intersection: if $S \subseteq \mathcal{T}$ and $|S| < \infty$, then
$$\bigcap_{U \in S} U \in \mathcal{T}.$$
That is, the intersection of finitely many elements in $\mathcal{T}$ is an element of $\mathcal{T}$.

Exercise 1. Let $A = \{a, b, c, d\}$. For each set $\mathcal{T}$, determine if $\mathcal{T}$ is a topology on A. If $\mathcal{T}$ is not a topology, find the smallest set containing $\mathcal{T}$ that is a topology.

a. $\mathcal{T} = \{\varnothing, A\}$

b. $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, A\}$

c. $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, A\}$

d. $\mathcal{T} = \{\varnothing, \{a, b\}, \{c, d\}, A\}$

e. $\mathcal{T} = \{\varnothing, \{a\}, \{a, b, c\}, \{c, d\}, A\}$

Exercise 2. Let A be a set. Prove that $\mathscr{P}(A)$ is a topology on A.

Remark. The definition of topology is often a difficult definition to satisfy because it is a lot of work to check that every union and every finite intersection is contained in the topology.

Definition. Let A be a set, and let $\mathcal{B}$ be a set of subsets of A. The set $\mathcal{B}$ is a *basis for a topology on A* if it satisfies the following properties.

(1) For each $x \in A$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$, and $B_3 \subseteq B_1 \cap B_2$.

Exercise 3. Let $A = \{a, b, c, d\}$. For each set $\mathcal{B}$, determine if $\mathcal{B}$ is a basis for a topology on A. If $\mathcal{B}$ is not a topology, explain why it violates the definition.

a. $\mathcal{B} = \{A\}$

b. $\mathcal{B} = \{\{a\}, \{a, b\}, \{a, b, c\}, A\}$

c. $\mathcal{B} = \{\{a\}, \{b\}, \{c, d\}\}$

d. $\mathcal{B} = \{\{b, c\}, \{a, b, c\}, \{b, c, d\}\}$

e. $\mathcal{B} = \{\{a\}, \{a, b, c\}, \{c, d\}\}$

Exercise 4. Prove that $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$.

Exercise 5. Prove that $\mathcal{B} = \{U \subseteq \mathbb{R} : |\overline{U}| < \infty\}$ is a basis for a topology on $\mathbb{R}$.

Definition. Let A be a set, and let $\mathcal{B}$ be a basis for a topology on A. The *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is the set of all $U \subseteq A$ that satisfy: if $x \in U$, then there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. That is,

$$\mathcal{T} = \{U \subseteq A : \text{for each } x \in U, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \text{ and } B \subseteq U\}.$$

Exercise 6. Prove that if A is a set, $\mathcal{B}$ is a basis for a topology on A, and $\mathcal{T}$ is the topology generated by $\mathcal{B}$, then $\mathcal{T}$ is a topology on A.

Exercise 7. Prove that if A is a set and $\mathcal{T}$ is a topology on A, then there exists a basis for a topology $\mathcal{B}$ that generates $\mathcal{T}$.

3.16 Open and closed sets

Some of the biggest challenges for students are keeping a working example in mind and keeping the notation straight.

MATH 2001
OPEN AND CLOSED SETS

Upcoming deadlines:

- Due Friday, Mar 18: final draft of proof 9.
- Due Wednesday, Mar 30: final draft of proof 10, first draft of proof 11.

Definition. Let A be a set, let $\mathcal{B}$ be a basis for a topology on A, and let X be a subset of A. The set X is *open* if for each $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq X$.

Definition. Let $\mathcal{B}$ be a basis for a topology on A, and let X be a subset of A. The set X is *closed* if $A - X$ (the complement of X in A) is open.

Exercise 1. Let $\mathcal{B} = \{(x - \epsilon, x + \epsilon) : \epsilon \in \mathbb{R}, \epsilon > 0\}$. (The topology which this basis generates is known as the *standard topology on* $\mathbb{R}$.) For each of the following sets, determine whether the set is open, closed, both, or neither. (Assume $a, b \in \mathbb{R}$.)

a. (a, b)	c. $(-\infty, b]$	e. $\mathbb{Z}$	g. $\varnothing$
b. $[a, b)$	d. $[a, b]$	f. $\mathbb{R}$	h. $\{10^{-n} : n \in \mathbb{N}\}$

Exercise 2. Let $\mathcal{B} = \{U \subseteq \mathbb{R} : |\overline{U}| < \infty\}$. (This is a basis for the *finite complement topology on* $\mathbb{R}$.) Repeat the previous exercise with this topology.

Exercise 3. Let $\mathcal{B} = \{[x, y] \subseteq \mathbb{R} : x, y \in \mathbb{R}\}$. (This basis for the *discrete topology on* $\mathbb{R}$.) Repeat the previous exercise with this topology.

3.17 Relations

Returning to Hammack: Chapter 11. This is 11.1–11.3. I do define equivalence relations and equivalence classes here, despite the title of this and the next worksheet.

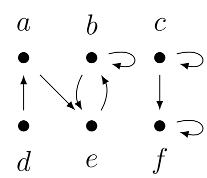
MATH 2001
RELATIONS

Definition 1. Let A be a set. The set R is a *relation* on A if $R \subseteq A \times A$.

Definition 2. Let A be a set and let R be a relation on A. Then

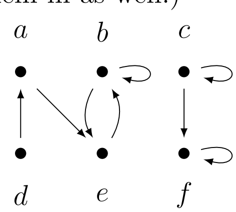
- R is *reflexive* if $(a, a) \in R$ for each $a \in A$;
- R is *symmetric* if $(a, b) \in R$ implies that $(b, a) \in R$;
- R is *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.

Example 3. Let $A = \{a, b, c, d, e, f\}$ and consider the relation R given by the following diagram.

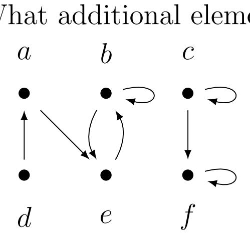


a.) What is R? (Write out the elements of R in set notation.)

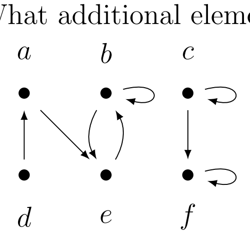
b.) What additional elements would have to be included for R to be reflexive? (Feel free to draw them in as well.)



c.) What additional elements would have to be included for R to be symmetric?



d.) What additional elements would have to be included for R to be transitive?



Example 4. Each of the following is a relation on $\mathbb{Z}$: $=, \neq, \leq, |, \nmid$. For each relation, determine whether it is reflexive, symmetric, and/or transitive. If a relation does not have a particular property, give an example illustrating why not.

Definition 5. A relation R is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 6. Which of the relations in Example 4 are equivalence relations?

Definition 7. Let A be a set, and let R be an equivalence relation on A. For any $a \in A$, the *equivalence class containing* a (denoted by $[a]$) is the set of all elements in A that are related to a. That is,

$$[a] = \{b \in A : (a, b) \in R\}.$$

Example 8. Let $A = \mathbb{Z}$, and let R be the relation on A defined by

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b = 2c \text{ for some } c \in \mathbb{Z}\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.

Example 9. Let $A = \mathbb{Z}^2 - \{(0, 0)\}$, and let R be the relation on A defined by

$$R = \{((a, b), (c, d)) : ad - bc = 0\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.

3.18 Equivalence relations

MATH 2001
EQUIVALENCE RELATIONS

Homework. Book exercises: Due Wednesday, April 13.
Section 11.0: 5, 7.
Section 11.1: 2, 8.
Section 11.2: 2, 4, 8.
Section 11.3: 4.

Proofs.
Friday, April 8: first draft of Proof 13.
Monday, April 11: final draft of Proof 12.
Wednesday, April 13: final draft of Proof 13 and first draft of Proof 14.

Example 1. Let $A = \{1, 2, 3, 4, 5, 6\}$, and let R be an equivalence relation on A defined by

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (4, 6), (6, 4), (4, 5), (5, 4), (5, 6), (6, 5)\}.$$

List the equivalence classes of R.

Example 2. Let R be an equivalence relation on A, where $A = \{a, b, c, d, e\}$. Suppose that $(a, d) \in R$ and $(b, c) \in R$. Write out the elements of R, and draw the graph of R.

Example 3. Let R be the relation on $\mathbb{Z}$ defined by

$$R = \{(a, b) : a, b \in \mathbb{Z}, 3a - 5b \text{ is even}\}.$$

Describe the equivalence classes.

Theorem 4. *Suppose that R is an equivalence relation on a set A, and suppose also that $a, b \in A$. Then $[a] = [b]$ if and only if $(a, b) \in R$.*

Proof.

$\square$

Definition 5. A *partition* of a set A is a set of non-empty subsets of A such that the union of all the subsets is equal to A, and the sets are pairwise disjoint. That is, if X and Y are in the partition, then $X \cap Y = \varnothing$.

Example 6. Find all the partitions of $A = \{a, b, c\}$.

Theorem 7. *Let R be an equivalence relation on A. Then $\{[a] : a \in A\}$ is a partition of A.*

Partitions are introduced at the end of this sheet.

3.19 Modular arithmetic

Formerly I used this as an introduction to group theory. This year, no time, so I cut out any mention of groups.

MATH 2001
MODULAR ARITHMETIC

Due Wednesday, April 20.

Book exercises: Section 11.4: 6, 7.

Proofs: Final draft of Proof 14 and first draft of Proof 15.

Definition. Let a, b, and n be integers. We say that a is *congruent to b modulo n* if $n \mid (a - b)$, and we write $a \equiv b \pmod{n}$. (TeX: `a \equiv b \pmod n`)

Exercise 1. Prove that *congruence modulo n* is an equivalence relation.

Exercise 2. The *division algorithm* states that if a and n are integers, then there exist unique integers q and r such that $a = qn + r$ and $0 \le r < n$.

Prove that $a \equiv r \pmod{n}$.

As a consequence, a and b have the same remainders when divided by n if and only if $a \equiv b \pmod{n}$. Since there are exactly n remainders when dividing by n (they are: $0, 1, 2, 3, \dots, n-1$), there are exactly n equivalence classes modulo n: $[0], [1], [2], \dots, [n-1]$.

Exercise 3. Write out the equivalence classes modulo 4 explicitly.

$[0] = \{ \qquad \}$ $\qquad [1] = \{ \qquad \}$

$[2] = \{ \qquad \}$ $\qquad [3] = \{ \qquad \}$

Exercise 4. We define the sum of equivalence classes as follows:

$$[a] + [b] = \{x + y : x \in [a], y \in [b]\}.$$

At the moment, there is no reason that the set on the left should be an equivalence class, but it turns out that is it.

Working modulo 3, write out the following sets explicitly.

$[0] + [0] = \{ \qquad \}$ $[0] + [1] = \{ \qquad \}$

$[1] + [1] = \{ \qquad \}$ $[1] + [2] = \{ \qquad \}$

Exercise 5. Write out the addition table for the integers modulo 4 and modulo 5. (Put the appropriate class in each box.)

+	[0]	[1]	[2]	[3]
[0]				
[1]				
[2]				
[3]				

+	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

Give a conjecture: $[a] + [b] = [\qquad]$ (what class?) Can you prove your conjecture?

Exercise 6. We can do the same for multiplication. In this case, we will simply define $[a]\cdot[b] = [a\cdot b]$. Fill out the multiplication tables for 4 and 5.

$\cdot$	[0]	[1]	[2]	[3]
[0]				
[1]				
[2]				
[3]				

$\cdot$	[0]	[1]	[2]	[3]	[4]
[0]					
[1]					
[2]					
[3]					
[4]					

Exercise 7. We all know that $x^2 = 1$ has two solutions in $\mathbb{R}$ (they are $x = 1$ and $x = -1$). How many solutions are there to $x^2 \equiv 1 \pmod{n}$ when $n = 4$? $n = 5$? $n = 8$? $n = 16$? $n = 24$?

3.20 Functions

Sections 12.1, 12.2, and 12.4 of Hammack.

MATH 2001
FUNCTIONS

Due Wednesday, April 20.

Book exercises: Section 11.4: 6, 7.

Proofs: Final draft of Proof 14 and first draft of Proof 15.

Due Wednesday, April 27.

Book exercises: Section 12.1: 2, 5, 8, 12. Section 12.2: 8, 14.

Definition. Let A and B be sets. We say that R is a *relation from A to B* if $R \subseteq A \times B$.

Definition. Let A and B be sets, and let f be a relation from A to B. The relation f is a *function from A to B* (written $f\colon A \to B$) if for each $a \in A$, the relation f contains exactly one element of the form (a, b).

Since (a, b) is unique to a, we write $f(a) = b$.

Definition. Let $f\colon A \to B$. The *domain* of f is A, and the *codomain* of f is B. The *image* (or *range*) of f is the set

$$\operatorname{im}(f) = \{b \in B : (a, b) \in f\}.$$

Definition. A function $f\colon A \to B$ is *injective* (or *one-to-one*) if whenever $f(a) = f(b)$, then $a = b$. Equivalently, if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$.

Definition. A function $f\colon A \to B$ is *surjective* (or *onto*) if $\operatorname{im}(f) = B$. That is, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Definition. A function f is *bijective* if f is both injective and surjective.

Exercise 1. Give an example of a function that is

a.) injective but not surjective;

b.) surjective but not injective;

c.) neither injective nor surjective;

d.) bijective.

Exercise 2. Prove that $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x$ is injective but not surjective.

Exercise 3. Prove that $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $f(x) = |x|$ is surjective but not injective. (Here $\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} : x \geq 0\}$.)

Exercise 4. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are injective functions, then $g \circ f\colon A \to C$ is injective.

Exercise 5. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are surjective functions, then $g \circ f\colon A \to C$ is surjective.

3.21 Inverses

Sections 11.5 and 11.6.

MATH 2001
INVERSES

Wednesday, April 20: first draft of Proof 15.

Monday, April 25. Book exercises Section 11.4: 6, 7. Final draft of Proof 14.

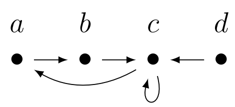
Wednesday, April 27. Book exercises: Section 12.1: 2, 5, 8, 12. Section 12.2: 8, 14. Section 12.5: 3, 6, 8.

Definition. The set R is a relation from the set A to the set B if ...

Definition. Let R be a relation from A to B. The *inverse relation* (denoted R^{-1}) is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Exercise 1. Let $A = \{a, b, c, d\}$ and consider the relation R from A to itself given by the following diagram. Write the sets R and R^{-1} (using the proper notation).



Definition. Suppose A and B are sets, and f is a relation from A to B. The relation f is a *function* if ...

Exercise 2. Let $A = \{n \in \mathbb{Z} : |n| \leq 2\}$, and let $f\colon A \to \mathbb{R}$ be defined by $f(x) = x^2$. Write the sets f and f^{-1}. Is f^{-1} a function? Explain why or why not.

Definition. Let $f\colon A \to B$ be a relation and suppose $U \subseteq A$. Then the *image of* U in B is the set

$$f(U) = \{f(x) \in B : x \in U\}.$$

Definition. Let $f\colon A \to B$ be a function and suppose $V \subseteq B$. Then the *inverse image of* V in A is the set

$$f^{-1}(V) = \{x \in A : f(x) \in V\}.$$

Exercise 3. Let f and A be defined as in Exercise 2. Write out each of the following sets:

a.) $f(A)$

b.) $f(\mathbb{N} \cap A)$

c.) $f^{-1}(\mathbb{R})$

d.) $f^{-1}(A)$

e.) $f^{-1}([1, 4])$

Exercise 4. Read the following theorem and the accompanying proofs.

Theorem 1. *Suppose $f\colon A \to B$ is a function, and let U and V be subsets of A. Then ...*

Proof. Suppose $x \in U \cup V$. Then $x \in U$ or $x \in V$.[a] If $x \in U$, then $f(x) \in f(U)$,[b] and so $f(x) \in f(U) \cup f(V)$.[c] Otherwise if $x \in V$, then $f(x) \in f(V)$,[d] so $f(x) \in f(U) \cup f(V)$.[e] Thus we have shown that if $x \in U \cup V$, then $f(x) \in f(U) \cup f(V)$. $\square$

Proof. Suppose $f(x) \in f(U) \cup f(V)$. Then $f(x) \in f(U)$ or $f(x) \in f(V)$.[f] If $f(x) \in f(U)$, then $x \in U$, and if $f(x) \in f(V)$, then $x \in V$.[g] In either case, $x \in U \cup V$, completing the proof. $\square$

1.) The authors of these proofs have given two rather different arguments.
   (a) What claim has the first author attempted to prove?

   (b) What claim has the second author attempted to prove?

2.) We will now check the validity of the arguments. For each marked line in the proofs above, cite the definition that justifies the statement... unless the statement is unjustifiable, then mark it with $*$.

(a)	(c)	(e)	(g)
(b)	(d)	(f)	

3.) For each starred statement in the previous question (you should have at least one starred statement), explain why the statement is incorrect.

   An error in a proof likely invalidates the proof. Are the claims in the proofs with errors still correct? If so, give a correct proof.

3.22 Induction

Now back to Chapter 10. I had not intended to leave induction until the very end of the course, and it seems like a bad practice to cover this topic in the last week of the semester. But that is just how things fell in to place for me. If you saw my comments in the topology section, I had intended to introduce induction back then, but I abandoned that plan in order to cover more of the content in the book.

MATH 2001
PROOF BY INDUCTION

General outline of a proof by induction.

Proof.
1. Base case: verify that the first statement is true.
2. Induction step: show that if the k-th statement is true, then the $(k+1)$-st statement is true.

$\square$

Example 1. Prove that $7 \mid 4^{3n} - 1$ for every non-negative integer n (i.e. $n = 0, 1, 2, 3, \ldots$).

Proof.

$\square$

Example 2. Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof.

$\square$

Example 3. Let $a_1, a_2, a_3, \ldots$ be a sequence of integers where $a_1 = 3$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$ for each integer $n \geq 4$. Prove that $1 \leq \frac{a_n}{a_{n-1}} \leq 2$ for each $n \geq 4$.

Proof.

$\square$

Example 4. Let $A_1, A_2, \ldots, A_n$ be sets. Prove that $\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$.

Proof.

$\square$

Suggested proofs:
Chapter 10 (induction): 6, 7, 18.
Chapter 12.6 (functions and inverses): 10, 11, 13.

3.23 Induction II

MATH 2001
PROOFS BY INDUCTION

Some of the problems from yesterday's sheet are included on this one.

Arithmetic.

1.) Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for every integer $n \geq 1$.

2.) Prove that $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for every integer $n \geq 0$.

3.) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers where $a_1 = 3$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for each integer $n \geq 4$. Prove that $1 \leq \dfrac{a_n}{a_{n-1}} \leq 2$ for each $n \geq 4$.

Sets.

4.) Suppose that $A_1, A_2, A_3, A_4, \ldots$ is an infinite sequence of non-empty, nested sets. That is,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

and each A_i is non-empty.

(a) Prove that $\bigcap_{i=1}^{N} A_i$ is non-empty for every $N \in \mathbb{N}$.

(b) Is the set $\bigcap_{i=1}^{\infty} A_i$ empty or non-empty?

If possible, give an example where $\bigcap_{i=1}^{\infty} A_i$ is empty.

If possible, give an example where $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

5.) (a) Prove the finite versions of de Morgan's laws by induction.

(i) $\displaystyle\overline{\bigcap_{1\leq i\leq N} A_i} = \bigcup_{1\leq i\leq N} \overline{A_i}.$ (ii) $\displaystyle\overline{\bigcup_{1\leq i\leq N} A_i} = \bigcap_{1\leq i\leq N} \overline{A_i}.$

(b) Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of sets. Explain why induction cannot be used to prove the next two statements.

(i) $\displaystyle\overline{\bigcap_{i\in\mathbb{N}} A_i} = \bigcup_{i\in\mathbb{N}} \overline{A_i}.$ (ii) $\displaystyle\overline{\bigcup_{i\in\mathbb{N}} A_i} = \bigcap_{i\in\mathbb{N}} \overline{A_i}.$

Prove these "infinite" versions of de Morgan's laws using a method other than induction. In fact, these statements are true for arbitrary unions and intersections. Your proofs for (i) and (ii) should also work for (iii) and (iv). (You don't need to write the proofs twice.)

(iii) $\displaystyle\overline{\bigcap_{\lambda\in\Lambda} A_i} = \bigcup_{\lambda\in\Lambda} \overline{A_i}$ (iv) $\displaystyle\overline{\bigcup_{\lambda\in\Lambda} A_i} = \bigcap_{\lambda\in\Lambda} \overline{A_i}.$

Topology.

6.) (a) Prove that a finite intersection of open sets is open. That is, prove that $\bigcap_{i=1}^{N} A_i$ is open if each A_i is open (and $N \in \mathbb{N}$). (c.f. Proof 13)

(b) Give an example of an infinite intersection of open sets that is not open. (Hint: Proof ??)

(c) Prove that a finite union of closed sets is closed. That is, prove that $\bigcup_{i=1}^{N} A_i$ is closed if each A_i is closed. (Hint: de Morgan's laws.)

(d) Give an example of a infinite intersection of closed sets that is not closed.

7.) (a) Prove that a union of arbitrarily many open sets is open. That is, prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is open if each A_λ is open.

(b) Prove that the intersection of arbitrarily many closed sets is closed.

Functions.

8.) Let $f \colon A \to A$ be a bijection. Prove that $f^n \colon A \to A$ is a bijection. [Hint: functions worksheet.]

Here, f^n denotes the n-fold composition of f with itself. That is, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc. In general, $f^{n+1}(x) = f(f^n(x))$.

9.) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function with the property that $|f(b) - f(a)| < \frac{1}{2}|b-a|$ for all $a, b \in \mathbb{R}$. Prove that $|f^n(b) - f^n(a)| < \frac{1}{2^n}|b-a|$ for all $a, b \in \mathbb{R}$.

Critique the following proof.

Theorem. *All horses are the same color.*

Proof. We prove the result by induction by showing that in every collection of n horses, all the horses have the same color.

For the base case, let $n = 1$. If there is only one horse, then there is only one color (e.g. blue). Thus in every set of 1 horses, every horse in that set has the same color as all other horses in that set.

For the induction step, suppose there is an n for which in every collection of n horses, all the horses have the same color. Select any n of the horses on Earth, then as we now have n horses, all of these horses have the same color. For simplicity, let's say that they are all blue. Remove one horse from this collection (leaving $n-1$ blue horses) and add to this set any horse on Earth that was not previously selected. We now have a new set of n horses, and therefore they all have the same color. In particular, the newly selected horse is blue as it has the same color as the other $n-1$ horses, which we already know to be blue. Bring back the blue horse that we initially removed, and we now have $n+1$ blue horses. This completes the induction step.

Finally, as the number of horses on Earth is finite, we can induct up to that number and conclude that all horses are the same color. $\square$

Chapter 4

Quizzes

All things quiz are located in the `Quizzes` folder.

Quizzes were given once per week (almost always first thing on Monday) with a few exceptions. Time allowed for quizzes varied—I did not set a time limit, I would just give a warning ("two more minutes to finish up") when most students were near completing the quiz. I would guess that a 'short' quiz lasted about 10 minutes at the start of the course and closer to 20 minutes near the end, while the longer quizzes (such as the group quizzes) went as long as a full class period.

The questions on my quizzes varied greatly, and so part of the time consideration was to allow students enough time to understand the (sometimes lengthy) instructions on the quiz.

Pdfs of the quizzes with solutions are included below. Dates on the quizzes may not be the date the quiz was given; the dates are the last date that the file was updated.

4.1 Sets

I think Question 3a on this quiz is an important type of question. This type of question only appears a few more times in my course, but it seems like a reasonable thing to ask whenever new notation is introduced.

MATH 2001
QUIZ 1

1.) (1 pt) Write your name in the top *right* corner of the page.

2.) (8 pts) Complete each statement so that is a precise definition.

i.) Two sets are *equal* if ...

they contain exactly the same elements.

ii.) A *finite set* is ...

a set containing finitely many elements.

iii.) If A is a finite set, the *cardinality* of A is ...

the number of distinct elements in A.

iv.) The set that contains zero elements is the ...

empty set.

3.) (7 pts) Let $A = \{-4, -2, 3, 5, 6\}$, and consider the expression:

$$B = \{x \in A : 3 \leq |x| \leq 6\}.$$

(a) In words, write out the expression in the line above. (What would you say if you had to read this expression out loud?)

The set B is the set of all x in A where the absolute value of x is greater than or equal to 3 and less than or equal to 6.

(b) Write out the set Y *explicitly* (using proper set notation).

$Y = \{-4, 3, 5, 6\}$.

(c) Compute $|A| + |B|$.

$|A| + |B| = 5 + 4 = 9$.

Date: May 4, 2016.

4.2 Subsets

I particularly like Question 4 on this quiz. It is a very efficient way of asking $\binom{6}{2}$ true/false questions. I liked it so much that I reused the idea on the final.

MATH 2001
QUIZ 2

1. (1 pt) Write your name in the top *right* corner of the page.
2. (6 pts) Answer each question with the precise definition of each highlighted term. Write your answer as a **complete sentence or sentences**. You may use set-builder notation, but the expression should still be incorporated as part of a complete statement.

   As a model for what I am looking for, I have started the first answer for you.

   i.) Suppose X and Y are sets. What does it mean for Y to be a *subset* of X?

   The set Y is a subset of X if ...

   every element of Y is an element of X.

   ii.) Suppose X is a set. What is the *power set* of X?

   The power set of X is the set of all subsets of X. In set builder notation,
   $$\mathscr{P}(X) = \{A : A \subseteq X\}.$$

   iii.) Suppose X and Y are sets. What is the *Cartesian product* of X and Y (in that order)?

   The Cartesian product $X \times Y$ is the set of ordered pairs where the first coordinate is an element of X and the second is an element of Y. That is,
   $$X \times Y = \{(a, b) : a \in X \text{ and } b \in Y\}.$$

3. (4 pts.) Suppose A is a finite set of cardinality n. Compute each of the following.

   (a) $|A^4| = n^4$

   (b) $|\mathscr{P}(A)| = 2^n$

   (c) $|\mathscr{P}(A^2)| = 2^{n^2}$

   (d) $|A^2 \times \mathscr{P}(\varnothing)| = n^2$

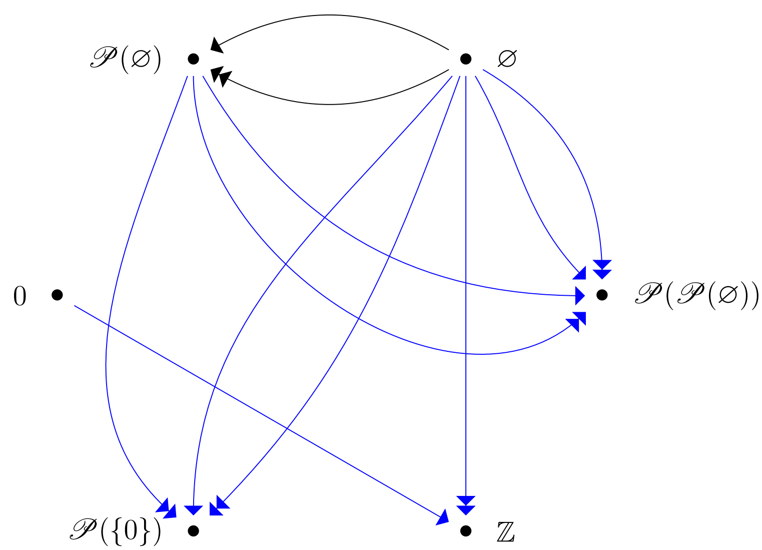
(Continued on reverse.)

Date: May 5, 2016.

4. (1 pt per correct arrow, −0.5 pt per incorrect arrow)

   - Draw a **single-headed arrow** from one set to the next if the first is an **element** of the second.
   - Draw a **double-headed arrow** if the first element is a **subset** of the second.

   As an example, I have drawn two arrows for you.



It can be helpful to write out some of these sets:

$$\mathscr{P}(\varnothing) = \{\varnothing\}$$
$$\mathscr{P}(\mathscr{P}(\varnothing)) = \{\varnothing, \{\varnothing\}\} = \{\varnothing, \mathscr{P}(\varnothing)\}$$
$$\mathscr{P}(\{0\}) = \{\varnothing, \{0\}\}.$$

4.3 Set operations

The very last question on this quiz (Question 4c) is a bit bungled. I wanted to combine a number of set operations, but I realized as I gave the quiz that a number of the intervals end up being empty sets. That is not necessarily a problem if the students know the definitions, but these were not definitions that I expected my students to know, and I was worried that students would assume that $[0, -2] = [-2, 0]$. So to avoid this complication, I attempted to adjust the problem mid quiz—hence 4c.ii. Alas, that did not solve the issue. I ended up just counting the problem as extra credit. In hind sight, I should have just put the definition for a closed interval on the board.

MATH 2001
QUIZ 3

1.) (1 pt) Write your name in the top *right* corner of the page.

2.) (8 pts) Give the precise definition of each of the following terms. Your answers must be a complete sentence (or sentences).

(a) Finite set

A *finite set* is a set that contains finitely many (distinct) elements.

(b) Union

Given two sets A and B, the *union* of A and B is the set of all elements that are in A or in B (or in both sets). In set builder notation,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

(c) Finite union

A *finite union* is the union of finitely many sets.

Given a finite list of sets, $A_1, A_2, \ldots, A_n$, the union of these sets is

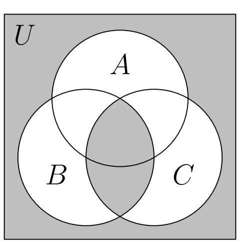
$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i \in \mathbb{Z}, \text{ where } 1 \leq i \leq n\}.$$

(d) Set difference

Given two sets A and B, the *difference* of A and B is the set of all elements that are in A but not in B. In set builder notation,

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

3.) (2 pts) Give an expression that describes the shaded region.

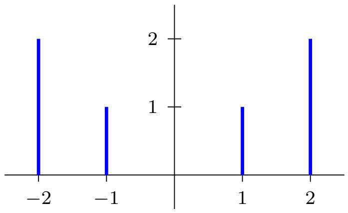


$\overline{A \cup B \cup C} \cup (B \cap C)$

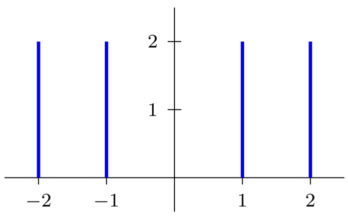
Date: May 4, 2016.

4.) (6 pts) Let $A_n = \{-n, n\}$, $B_n = [0, n]$, and $C_n = A_n \cup B_n$. Sketch each of the following sets in the xy-plane.

(a) $\displaystyle\bigcup_{n=1}^{2}(A_n \times B_n) = (\{-1,1\} \times [0,1]) \cup (\{-2,2\} \times [0,2]).$

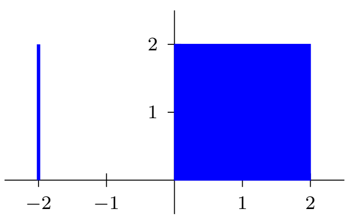


(b) $\displaystyle\left(\bigcup_{n=1}^{2} A_n\right) \times B_2 = (\{-1,1\} \cup \{-2,2\}) \times [0,2] = \{-2,-1,1,2\} \times [0,2].$



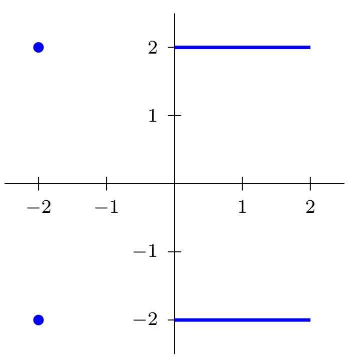
(c.i) $\displaystyle\bigcup_{n\in A_2}(C_n \times B_n) = ((\{-2,2\} \cup [0,-2]) \times [0,-2]) \cup ((\{-2,2\} \cup [0,2]) \times [0,2])$

$$= (\{-2\} \cup [0,2]) \times [0,2].$$



(c.ii) $\displaystyle\bigcup_{n\in A_2}(C_n \times A_n) = ((\{-2,2\} \cup [0,-2]) \times \{-2,2\}) \cup ((\{-2,2\} \cup [0,2]) \times \{-2,2\})$

$$= (\{-2\} \cup [0,2]) \times \{-2,2\}.$$



4.4 Arrange statements to give a proof about sets

The quiz asks the students to arrange statements to give a proof of the claim: $\mathscr{P}(A-B) \subseteq \mathscr{P}(A) - \mathscr{P}(B)$, where $\mathscr{P}(X)$ is the power set of X.

Unfortunately, this claim is false (an oversight on my part). If used again, the question should be reworded to prove that $\mathscr{P}(A) - \mathscr{P}(B) \subseteq \mathscr{P}(A-B)$, or (less ideally) to prove that $\mathscr{P}(A-B) - \{\varnothing\} \subseteq \mathscr{P}(A) - \mathscr{P}(B)$.

Another "fix" would be to use this as a proof analysis problem, like the proof on each of the next two quizzes.

MATH 2001
QUIZ 4

1. (1 pt) Write your name in the top *right* corner of the page.
2. (12 pts) Arrange the statements below to form the body of a proof. The statements should be arranged so that each statement follows logically from the preceding statement.

   Each sentence in the proof should begin with a lettered statement and end with a numbered statement. The lettered statements may be used more than once; each numbered statement should be used exactly once.

   Write the letter and numeral in the boxes provided below. (You do not have to write out the complete statements.)

   Proof. Suppose that $X \in \mathscr{P}(A - B)$.

Sentence	Letter	Numeral
(2)	A	V
(3)	B	IV
(4)	C	I
(5)	B	VI
(6)	A	III
(7)	C	II

↑ the first sentence is up here

Sketch of proof:

$X \in \mathscr{P}(A-B)$

$\Rightarrow X \subseteq A - B$ (def of power set)

$\Rightarrow$ if $y \in X$, then $y \in A - B$ (def of subset)

$\Rightarrow$ if $y \in X$, then $y \in A$ and $y \notin B$ (def of difference)

(i.e. if $y \in X$, then $y \in A$, and if $y \in X$, then $y \notin B$)

$\Rightarrow X \subseteq A$ and $X \not\subseteq B$ (def of subset)

$\Rightarrow X \in \mathscr{P}(A)$ and $X \notin \mathscr{P}(B)$ (def of power set)

$\Rightarrow X \in \mathscr{P}(A) - \mathscr{P}(B)$ (def of difference).

□

A. By the definition of *power set* ...

B. By the definition of *subset* ...

C. By the definition of *set difference* ...

I. ... $y \in A$ and $y \notin B$.

II. ... $X \in \mathscr{P}(A) - \mathscr{P}(B)$.

III. ... $X \in \mathscr{P}(A)$ and $X \notin \mathscr{P}(B)$.

IV. ... if $y \in X$, then $y \in A - B$.

V. ... $X \subseteq A - B$.

VI. ... since $y \in X \Rightarrow y \in A$, we see that $X \subseteq A$, and since $y \in X \Rightarrow y \notin B$, we have $X \not\subseteq B$.

(*continued on back*)

Date: May 4, 2016.

3. (2 pts) The proof on the front is a proof of what? Give an answer that does not involve X or y.

   The argument is a proof that $\mathscr{P}(A - B) \subseteq \mathscr{P}(A) - \mathscr{P}(B)$.

4. (2 pts) The proof on the front lacks an introduction. Write a short introduction for that proof (no more than three sentences) that explains what will be proved and how it will be proved.

   Suppose that A and B are sets. We prove that $\mathscr{P}(A - B) \subseteq \mathscr{P}(A) - \mathscr{P}(B)$ by showing that if $X \in \mathscr{P}(A - B)$, then $X \in \mathscr{P}(A) - \mathscr{P}(B)$.

Bonus: There is an error in proof/claim; +3 points if you can find and correct the error. (Either correct the proof, or correct the claim.)

4.5 Prove a statement about sets (group quiz)

Let B, C, and D be sets. Prove that if $C \neq \varnothing$ and $B \times C \subseteq C \times D$, then $B \subseteq D$.

The proof of this statement is longer than any argument that the students have had to give up to this point. Nor is this proof as straightforward as the proofs that they had previously encountered. The class environment during this quiz was very much like when the students had a worksheet. I was quite active answering questions throughout the class period. It took about 40 minutes for the groups to finalize their proofs. This group assignment was immediately following up with an individual quiz (the next quiz).

MATH 2001
QUIZ 5 INSTRUCTIONS

Work in groups of up to three people to give a complete proof for the following theorem.

Theorem. *Let B, C, and D be sets. If $C \neq \varnothing$ and $B \times C \subseteq C \times D$, then $B \subseteq D$.*

Points are awarded as follows:

- (1 pt) Name(s) in the top right corner.
- (1 pt) Writing is neat and legible.
- (1 pt) Each relevant and correctly stated definition.
- (7 pts) Complete sketch of the proof (optional, but recommended).
- (Remaining points, up to 14) Complete proof: introductory statements, all statements are complete sentences, every statement is justified appropriately, etc.

This quiz is out of 16 points. So, for example, if you state two relevant definitions and sketch a complete outline, then I will grade your written proof out of 4. If you just give me a proof (no definitions, no outline), then I will grade the proof out of 14.

You may use the space below for scratch work.

Write your final proof on the other worksheet.

Date: February 23, 2016.

MATH 2001
QUIZ 5

1. Write your name(s) in the top right corner.
2. Write is neatly and legibly.
3. Prove the following theorem.

Theorem. *Let B, C, and D be sets. If $C \neq \varnothing$ and $B \times C \subseteq C \times D$, then $B \subseteq D$.*

There are several approaches to proving this statement, so I will give two arguments.

Proof. Let B, C, and D be sets, and suppose that $B \times C \subseteq C \times D$ and $C \neq \varnothing$. We prove that $B \subseteq D$ by showing that if $x \in B$, then $x \in D$.

Suppose $x \in B$, and since $C \neq \varnothing$, let $y \in C$. Then by the definition of Cartesian product, $(x, y) \in B \times C$. Therefore, $(x, y) \in C \times D$ by the definition of subset and since $B \times C \subseteq C \times D$. Thus by the definition of Cartesian product, we conclude that $x \in C$ and $y \in D$.

In particular, we now have $x \in B$ and $x \in C$, and so $(x, x) \in B \times C$. So by a similar argument, $(x, x) \in C \times D$, and therefore $x \in D$. Thus we have shown that if $x \in B$, then $x \in D$, whence $B \subseteq D$ by the definition of subset. $\square$

I'll now give a second argument, which judging by class today, will likely be a closer match to the proof your group wrote.

Proof. Let B, C, and D be sets, and suppose that $B \times C \subseteq C \times D$ and $C \neq \varnothing$. We prove that $B \subseteq D$ in two steps: we prove that $B \subseteq C$ and $C \subseteq D$, and from this we conclude that $x \in B \Rightarrow x \in C$.

We will prove that $B \subseteq C$ and $C \subseteq D$ simultaneously by showing that $x \in B \Rightarrow x \in C$ and $y \in C \Rightarrow y \in D$.

Suppose that $x \in B$ and $y \in C$. Then $(x, y) \in B \times C$ by the definition of Cartesian product. Since $B \times C \subseteq C \times D$, we have $(x, y) \in C \times D$ by the definition of subset. Therefore $x \in C$ and $y \in D$ by the definition of Cartesian product. Thus we have shows that $x \in B \Rightarrow x \in C$ and $x \in C \Rightarrow x \in D$. So by the definition of subset, $B \subseteq C$ and $C \subseteq D$.

To finish off the proof that $B \subseteq D$, suppose that $x \in B$. Then $x \in C$ since $B \subseteq C$, and so $x \in D$ since $C \subseteq D$. Thus $B \subseteq D$ by the definition of subset. $\square$

Remark. Why is it important that $C \neq \varnothing$? Well, because the theorem is false if $C = \varnothing$, and that is a quiz question. But what about the other sets, B and D?

If $B = \varnothing$, then the conclusion of this theorem is automatically satisfied: if $B = \varnothing$, then $B \subseteq D$ since $\varnothing$ is a subset of every set.

What if $D = \varnothing$? The theorem holds, but this is a great question which will take a bit of work to resolve. The argument requires a proof by contradiction (which we will see later this semester), so I won't give it here, but the basic gist is as follows. If $D = \varnothing$, then $B = \varnothing$, and so we are back in the previous case.

Since these cases weren't addressed in the proofs above, are the proofs complete? No. These cases where $B = \varnothing$ and $D = \varnothing$ should be addressed in the proofs as well. In hindsight, I should have stipulated that all of the sets are non-empty to avoid these extraneous cases.

Edit: Actually, $D = \varnothing$ is not a concern. If $C \neq \varnothing$, then $D \neq \varnothing$.

4.6 Arrange statements to give a proof about sets

This quiz was given immediately following the previous group quiz.

The idea here was to have the students work though a harder proof as a group (on the previous quiz), and with those thoughts in mind, write a proof of a simpler statement on their own. Originally, I had lots of followup questions (along the lines of Question 1) as a check on how involved each student was on the group portion. However, after thinking about the time considerations, I ended up stripping the quiz down and providing most of the structure for the proofs.

MATH 2001
QUIZ 6

1. (1 pt) Write your name in the top *right* corner of the page.
2. (2 pts) You proved: if $C \neq \varnothing$ and $B \times C \subseteq C \times D$, then $B \subseteq D$. Why is it necessary that $C \neq \varnothing$? Fill in the blanks with a concrete example that illustrates why the theorem is false when $C = \varnothing$.

   Any choice of B and D where $B \nsubseteq D$ is a correct answer for this problem.

   If $B = \{1\}$, $D = \{2\}$, and $C = \varnothing$. Then $B \times C = \varnothing$, and $C \times D = \varnothing$, but $B \nsubseteq D$.
3. Sketch a proof for the statement: if $B \subseteq C$, then $A \times B \subseteq A \times C$.
   (a) (2 pts) Write a one or two sentence introduction for the proof of this statement.

   Suppose A, B, and C are sets and that $B \subseteq C$. We prove that $A \times B \subseteq A \times C$ by showing that if $(x, y) \in A \times B$, then $(x, y) \in A \times C$.

   (b) (4 pts) Arrange the statements to give an outline for the body of the proof. Justify each implication in the space after each line (e.g. cite a definition).

   $a \Rightarrow c$ (definition of Cartesian product)
   $\Rightarrow d$ (definition of subset, and $B \subseteq C$ (given))
   $\Rightarrow b$ (definition of Cartesian product)

   a. $(x, y) \in A \times B$
   b. $(x, y) \in A \times C$
   c. $x \in A$ and $y \in B$
   d. $x \in A$ and $y \in C$

4. (0.5 pts per blank) Fill in the blanks to complete a proof of the following statement: if $C \neq \varnothing$ and $A \times C \subseteq B \times C$, then $A \subseteq B$.

Proof. Let A, B, and C be sets, where $A \times C \subseteq B \times C$ and $C \neq \varnothing$. We prove that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

Suppose $x \in A$. Since $C \neq \varnothing$, the set C contains at least one element; call that element y. Therefore, since $x \in A$ and $y \in C$, we know that $(x, y) \in A \times C$ by the definition of Cartesian product. So, $(x, y) \in B \times C$ since $A \times C \subseteq B \times C$. Hence, by the definition of Cartesian product, we see that $x \in B$ and $y \in C$. Thus we have shown that if $x \in A$, then $x \in B$, and therefore $A \subseteq B$ by the definition of subset. $\square$

Date: March 2, 2016.

4.7 Prove a statement about sets (group quiz)

Prove that $\{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$ is equal to the set of even integers.

On this group assignment, I made the outline and definitions necessary. In part, this was so that more than one person in the group had to do some writing. This also provides a bit more separation between the logic and the writing.

This quiz took more than one class period. One direction of the proof is easy. I spent quite a bit of time with each group trying to motivate how they would address the reverse inclusion.

MATH 2001
QUIZ 7

Work in groups of up to three people to give a complete proof of the following statement.

Problem. Prove that $\{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$ is equal to the set of even integers.

Points are awarded as follows:

- (1 pt) Name(s) in the top right corner.
- (1 pt) Writing is neat and legible.
- (1 pt) Each relevant and correctly stated definition.
- (7 pts) Complete sketch of the proof.
- (7 pts) Complete proof: introductory statements, all statements are complete sentences, every statement is justified appropriately, etc.

You may use the space below for scratch work, write your outline and proof on separate sheets.

Date: March 4, 2016.

MATH 2001
QUIZ 7 – Outline

1. Write your name(s) in the top right corner.
2. Write is neatly and legibly.
3. State the definitions cited in your proof.
4. Provide a complete outline for a proof of the following claim.

Problem. Prove that $\{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$ is equal to the set of even integers.

Definitions:

Definition (Even)**.** An integer a is *even* if $a = 2c$ for some $c \in \mathbb{Z}$.

Theorem (Double containment)**.** *If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Definition (Subset)**.** If A and B are sets, then $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

Outline:

Let $A = \{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$ and $B = \{2c : c \in \mathbb{Z}\}$.

To prove that $A = B$, prove that $A \subseteq B$ and $B \subseteq A$.

($A \subseteq B$):

$$\begin{aligned} x \in A &\Rightarrow x = 18n + 8m \quad \text{for some } n, m \in \mathbb{Z} \\ &\Rightarrow x = 2(9n + 4m) \\ &\Rightarrow x = 2c \quad \text{where } c = 9n + 4m \in \mathbb{Z} \\ &\Rightarrow x \in B. \end{aligned}$$

($B \subseteq A$):

$$\begin{aligned} x \in B &\Rightarrow x = 2c \quad \text{for some } c \in \mathbb{Z} \\ &\Rightarrow x = (18 + 8(-2))c \\ &\Rightarrow x = 18c + 8(-2c) \\ &\Rightarrow x = 18n + 8n \quad \text{where } n = c \in \mathbb{Z} \text{ and } m = -2c \in \mathbb{Z} \\ &\Rightarrow x \in A. \end{aligned}$$

MATH 2001
QUIZ 7 – Proof

1. Write your name(s) in the top right corner.
2. Write is neatly and legibly.
3. Provide a complete proof of the following claim.

Problem. Prove that $\{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$ is equal to the set of even integers.

Proof. Let $A = \{18n + 8m : n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}\}$, and let B be the set of even integers. We prove that $A = B$ by showing that $A \subseteq B$ and $B \subseteq A$.

($\subseteq$) We start by proving that $A \subseteq B$ by showing that if $x \in A$, then $x \in B$.

Suppose that $x \in A$. Then $x = 18n + 8m$ for some $n, m \in \mathbb{Z}$. Hence $x = 2(9n + 4m) = 2c$, where $c = 9n + 4m \in \mathbb{Z}$, and thus x is even by definition. Therefore, $x \in B$, and thus $A \subseteq B$ by the definition of subset.

($\supseteq$) We now prove that $B \subseteq A$ by showing that if $x \in B$, then $x \in A$.

Suppose that $x \in B$, that is, suppose that x is even. Then $x = 2c$ for some $c \in \mathbb{Z}$. Note that

$$\begin{aligned} x &= 2c \\ &= (18 + 8(-2))c \\ &= 18c + 8(-2c) \\ &= 18n + 8m, \end{aligned}$$

where $n = c$ and $m = -2c$. Since c is an integer, so are m and n, hence $x \in A$. Thus, by the definition of subset, $B \subseteq A$.

As $A \subseteq B$ and $B \subseteq A$, we have proven that $A = B$. $\square$

4.8 Analyze a proof

Perhaps it is better to do this with an errorless proof rather then the mess I gave my students. Originally, I was going to give my students two variations of this proof, and have the students comment on both proofs. There are better ways of doing this type of assessment, and other than seeing how the students reacted to this writing, it is probably more effective to give this as a worksheet rather than a quiz.

MATH 2001
QUIZ 8

On this quiz, you are tasked to analyze and assess a proof.

INSTRUCTIONS

Task 1. [First glance.] Give the argument a *quick* once-over. On this first pass, you do **not** need to understand the proof. **Do not get caught up in the details.**

For each sentence, ask the question: "do I understand, or could I work out, the meaning of this statement?"

- If 'yes' ("this statement seems reasonable enough"), put a ✓ by the sentence number.
- If 'no' ("What? Something seems wrong. I'm confused."), put a **?** above the point(s) in the sentence that you find confusing, and **move on** to the next sentence.

Task 2. [Understand.] Read through the proof a second time with an eye towards understanding the argument as a whole.

Confusion often arrises from poorly worded sentences or inadequately justified statements.

- Attempt to resolve the confusions from the first reading (though it is fine if you cannot).

  Make small edits to the proof to aid your understanding of what is written.

(You may be tempted to rewrite the whole sentence, or even the entire argument. In the interest of time, I encourage you to not do this.)

Task 3. [What is being proved?] In the proof, the statement that the author is proving is not made explicit. We would like to recover that statement: "Given ____________, prove that ____________."

- Circle any information which you believe to be given in the statement of the problem and mark it with a "G".
- The author claims that they will show "if $Z \in C$ and $Z \in$ A." What will they have proven if they achieve this statement?

  Using the answers to these last two questions, write a statement of the form

  "Given ____________, prove that ____________"

  above the proof.

Task 4. [Is the argument sound?]

- Is the argument logically sound? You may want to sketch an outline of the proof.

Task 5. [Grade]

- At the bottom of the proof, give a grade based solely on the quality of writing.
- At the bottom of the proof, give a grade based solely on the logical accuracy of argument.

Date: March 17, 2016.

Quiz 8 proof

Proof. [1] A set is a collection, of some elements, and so are B and C . [2] Set $C = \overline{B}$ where by definition $\overline{B} = \{X \notin B\}$ and $B = \overline{A}$ where $\overline{A} = \{Y \notin A\}$ both by the definition of complementary set. [3] Set $Z \in C$.

[4] We prove the result by showing if $Z \in C$ and $Z \in$A. [5] Since $Z \in \overline{B}, Z \notin B$ by definition and because $C = \overline{B}$. [6] On the the other hand since $Y \notin A$, $Y \in \overline{A}$, and $Y \in B$ because B and $\overline{A}$ are the same set.

[7] Therefore if $Z \notin Y$, than $Z \notin B$, $Z \notin \overline{\text{A}}$ and $Z \in$ A. [8] Thus $Z \in C$ and $Z \in$ A. [9] Proof over. □

Some questions to consider (but you don't need to answer).

a.) In sentence 2, is the first word ("set") a noun or a verb?

b.) In sentence 3, is Z a set, a subset, an element, or some combination of these (which ones)?

c.) In sentence 5, what is the definition to which the author refers?

d.) There are A's and there are A's. Did you consider the possibility that these were two different sets?

There isn't a solution to this quiz per say, so I will "simply" offer my comments.

Billiam's proof

The name of the author has been changed so as to protect their identity.

Proof. [1] A set is a collection, of some elements, and so are B and C .

This sentence is silly; I almost don't know what to say.

- The first comma is wrong. [remove]
- There is a space before the period. [remove]
- The sentences starts out as the definition of a set... and then B and C are also sets? The confusion is compounded by the fact that Billiam never formally declares that A (or A?) is a set, but maybe that's what they are doing at the start of this sentence: "A is a set, and so are B and C"? And somehow Billiam is also trying to jam the definition into this sentence? In any case, this is a poor way of saying, "let $(A?)$ B and C be sets." [rewrite]

[2] Set $C = \overline{B}$ where by definition $\overline{B} = \{X \notin B\}$ and $B = \overline{A}$ where $\overline{A} = \{Y \notin A\}$ both by the definition of complementary set.

This is a common issue: stating a definition or explaining notation in the midst of other details. The sentence is cumbersome and it becomes difficult to pick out the important information. Although it is often helpful to remind the reader of the meaning of a term, clunky sentences detract from the argument. In general, it is a much better practice to remind the reader of the terms elsewhere.

Suggestions:

"Let $C = \overline{B}$ and $B = \overline{A}$." (Just give the important information.)

"Let $C = \overline{B}$ and $B = \overline{A}$, where the bar denotes the complement of the set." (A friendly reminder of the notation in a way that doesn't disrupt the important information. The full definition of complement can be given elsewhere.)

- This sentence is poorly written for the reason listed above. [rewrite]
- The set notation is incorrect. This is neither set-builder notation, nor is it an explicit presentation of elements. [fix set notation]
- I don't think I've ever heard "complementary" used for sets (unlike *complementary angles* in geometry, say). "The set B is the complement of C", or "B and C are complements" is much more standard. [change to "definition of set complement"]

[3] Set $Z \in C$.

This is fine, though "suppose $Z \in C$" or "let $Z \in C$" are preferable due to the potential noun/verb ambiguity of "set".

[4] We prove the result by showing if $Z \in C$ and $Z \in$A.

Stating the 'result' would be helpful. Perhaps Billiam misunderstood the question and his method doesn't actually address the original question.

- "showing if $Z \in C$ and $Z \in A$" is a fragment: "showing if [this] and [that], then...". Or maybe this is just mis-worded. Without knowing the result, I don't know. [complete or reword sentence]
- TeX error at the end of line. [fix]

[5] Since $Z \in \overline{B}, Z \notin B$ by definition and because $C = \overline{B}$.

- Why is $Z \in \overline{B}$?

  This is an example of a disconnected thought as this line follows directly from the third: if $Z \in C$, then $Z \in \overline{B}$ because $C = \overline{B}$. The fourth sentence is out of place and breaks the logical flow of the argument. [swap sentences 3 and 4]

- What definition?

  Probably the definition of complement, but we have a lot of definitions. It's better to be explicit and avoid ambiguity.

- Is the comma an 'and' or a 'then'?

  Not such a big deal on this line, but it is a bit ambiguous: "since $Z \in \overline{B}$ and $Z \notin B$..." or "since $Z \in \overline{B}$, then $Z \notin B$..." [insert helping word]

[6] On the the other hand since $Y \notin A$, $Y \in \overline{A}$, and $Y \in B$ because B and $\overline{A}$ are the same set.

- Another common mistake: this is another big jump in logic. We went from $Z \notin B$ to $Y \notin A$ without explanation. Do not do this in your proofs.

  Consider if you did this with directions.

  > Okay Zorro, I'm going to tell you how to get from Denver to New York City. If you're in Denver, you can take I-70 to St. Louis. On the other hand, Yvonne is in New York City, and she can take the NJ Turnpike down to Philadelphia.

  What? What does Yvonne have to do with Zorro's mission? Keep talking to Zorro!

  [remove this sentence]

- Here's where the ambiguous comma issue really shows. If you don't want it to look like a list, put some words between the commas.
- Extra space before first comma.
- "The" is written twice at the start of the sentence.

[7] Therefore if $Z \notin Y$, than $Z \notin B$, $Z \notin \overline{\mathrm{A}}$ and $Z \in \mathrm{A}$.

- How can we determine if Z is or is not an element of Y? Probably $Z \neq Y$ is more appropriate since it appears that we are comparing elements in the sets A, B, and C. Even so...
- There are definite logical gaps here. Basically any justification for "if $Z \notin Y$, then $Z \notin B$" is wrong.
  - If $Z \notin Y$, then $Z \notin B$ only if $Y \subseteq B$. But we don't know whether $Y \subseteq B$ or not. So that's out.
  - If $Z \neq Y$, then it still could be that $Z \in B$.

  Oh wait, we already established that $Z \notin B$ back in sentence 5. Again I ask: why is Yvonne in this story?

- Commas are fine here. This is actually a list of things that we can conclude. Justification would be nice though. [justify statements]
- 'than' is the wrong word. [fix]

[8] Thus $Z \in C$ and $Z \in \mathrm{A}$.

Okay. At least we ended where Billiam said we would.

[9] Proof over.

This is a fragment. Also, if you follow the (now archaic? (too soon to call it 'archaic'?)) rule against ending a sentence with a preposition, there's that as well. (I guess I'm archaic.) [fix]

$\square$

Some questions to consider (but you don't need to answer).

a.) In sentence 2, is the first word ("set") a noun or a verb?

To me, it is a verb, but there is space for ambiguity.

b.) In sentence 3, is Z a set, a subset, an element, or some combination of these (which ones)?

Again, 'set' better be a verb (if not, this is a terribly worded sentence), so the only thing that this line is telling us is that Z is an element of the set C.

c.) In sentence 5, what is the definition to which the author refers?

Already addressed.

d.) There are A's and there are A's. Did you consider the possibility that these were two different sets?

No.

Summary

Billiam argued that if $Z \in C$, then $Z \in C$ and $Z \in A$, which answers the following prompt.

Given the sets A, B, and C, where $C = \overline{B}$ and $B = \overline{A}$, prove that $C \subseteq C \cap A$.

Admittedly, it is very odd to start with the assumption that $Z \in C$ and then conclude that $Z \in C$.

Grades

Quality of writing: Let's do a quick rundown. Out of the nine sentences, 3 and 8 are perfectly fine. Sentences 1, 2, and 6 are cumbersome head-scratchers, and lines 4 and 9 suffer from the grammatical deficiency of not being a sentence. That leaves 7, which has a typo, but otherwise works.

I give this one 6 fleebles.

Quality of logic Sentences 1: meh? Sentence 2 has set notation issues. Lines 3 and 8 are good. I have a tiff with 4, a spat with 5, a quarrel with 7, and 6 should be burned. Line 9: n/a.

The ideas are here and they are mostly in order, but they are interspersed with unnecessary detours. Not a great sign for demonstrating understanding of the subject. All in all, not too bad. Easy fix: cut out the bad parts.

Award: 7 schmeckles.

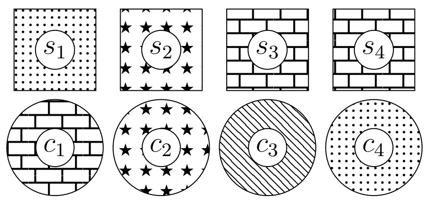
4.9 Logical statements and proof analysis

The second statement in question 1 is too troubling to manipulate and should not be used in the future.

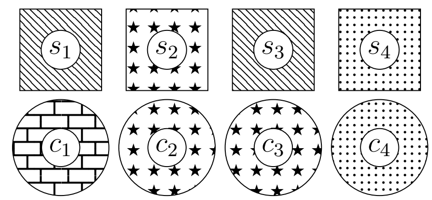
I am much more satisfied with the proof analysis portion of this quiz than the last, and I wish I had time to do more of these types of exercises during the semester.

MATH 2001
QUIZ 9

(1) **I.**



II.



Let $P(s_i)$ and $P(c_i)$ denote the pattern on the i-th square and circle, respectively.

- For each of the following statements, write its contrapositive, negation, and converse where requested.
- Additional, identify which diagrams satisfy each statement.

1. Satisfied by: None

   For each i, $P(c_i)$ is "bricks," but $P(s_i)$ is not.

   I, II — Negation: For all i, $P(c_i)$ is not "bricks" or $P(s_i)$ is "bricks".

2. II — If there exists an i such that $P(c_i) = P(s_i)$, then there exists a j such that $j \neq i$ and $P(c_j) = P(s_j)$.

   II — Contrapositive: If there exists an i such that for all j either $j = i$ or $P(c_j) \neq P(s_j)$, then $P(s_i) \neq P(c_i)$.

   I — Negation: There exists an i such that $P(c_i) = P(s_i)$, and for all j, either $j = i$ or $P(c_j) \neq P(s_j)$.

   II — Converse: If there exists a j such that $P(c_j) = P(s_j)$, then there exists an i such that $i \neq j$ and $P(s_i) = P(c_i)$.

Date: May 2, 2016.

(2) Edit the following proof for its writing only. We will tackle the logic on the next page.

- Fix all spelling and punctuation errors.
- Mark any grammatical errors. Replace or remove individual words so that the sentences are grammatically correct. Make as few changes as possible and avoid rewriting entire sentences (or the whole proof for that matter.)
- Correct all notational errors relating to mathematical symbols or definitions.

Question: Is $\sqrt{4}$ irrational? Prove your claim.

Proof. We prove (by contradiction) that $\sqrt{4}$ is irrational.

Assume $\sqrt{4}$ is rational , that is ,

$$\sqrt{4} = \frac{a}{b} , \quad \text{where } a, b \in \mathbb{Z} . \tag{1}$$

Assume ~~if~~ [further that] a does not share a common factor with b . ~~because~~ ~~$a \mid b$~~ [In other words, a/b] is a reduced fraction. Then by clearing denominators and squaring , equation (1) ~~equals~~ [is equivalent to]

$$a^2 = 4b^2 . \tag{2}$$

so a^2 is even , and [therefore a is even. Hence] ~~it~~ $a = 2c$, for some $c \in \mathbb{Z}$. ~~Rewording~~ [Substituting] ~~this~~ [$a = 2c$] into ~~the~~ equation [(2) yields]

$$4c^2 = 4b^2 , \tag{3}$$

so $c = \pm b$. ~~To get a contradiction see~~ [Note] that a and b share a common factor: c. Specifically , $c \mid b$ because $1 \cdot c = b$, and $c \mid a$ becuase $2 \cdot c = a$. [The fact that] $c \mid a$ and $c \mid b$ ~~which is a contradiction to the statement~~ ~~$a \mid b$~~ [contradicts the assumption that a/b] is reduced.

Thus ~~proofing~~ ~~that~~ $\sqrt{4}$ is irrational. $\square$

The majority of the errors are somewhat minor: word choice, sentence structure, spelling errors. But there are also a number of (what I would consider) significant mistakes.

- Failing to declare that a, b, and c are integers.
- Missing steps and lack of justification, particularly in the middle of the argument.
- Use of ambiguous terms: "*it* equals $2c$. Rewording *this* into *the equation*..."
- Mistaking $a \mid b$ for $\frac{a}{b}$.

All told, there are about eight significant errors and another 20 or so minor ones.

(3) *Proof style:* (circle one)   direct   contrapositive $\boxed{\text{contradiction}}$

Assumptions: On what assumptions is the author basing his/her argument? (What are the starting assumptions?)

The author assumes that $\sqrt{4}$ is rational, and that $\sqrt{4} = a/b$, where a/b is a fully reduced fraction.

Outline: For each line, cite the appropriate theorem, definition, etc. which justifies the step. If the step is the result of a simple algebraic manipulation, write "alg" for the justification. If the statement does not logically follow from the previous line, write "$*$" for the justification.

$\sqrt{4} = \frac{a}{b}$	$\Rightarrow$	$4b^2 = a^2$	(alg.)
	$\Rightarrow$	a^2 is even	(def. of even, $a^2 = 2(2b^2)$, assuming $a, b \in \mathbb{Z}$)
	$\Rightarrow$	a is even	(proved in class)
	$\Rightarrow$	$a = 2c$	((where $c \in \mathbb{Z}$) def. of even)
	$\Rightarrow$	$4b^2 = 4c^2$	(alg.)
	$\Rightarrow$	$b = c$	(alg. (actually, $b = \pm c$))
$a = 2c$	$\Rightarrow$	$c \mid a$	(def. of divides)
$b = c$	$\Rightarrow$	$c \mid b$	(def. of divides)
$c \mid a$ and $c \mid b$	$\Rightarrow\Leftarrow$		(a and b share a factor, so a/b is not reduced *)

Unjustified statements: For each "$*$", explain why this is a logical gap or gaffe. You do not have to correct the error, just explain why it is an error.

Although the argument is a bit sloppy and light on details, the argument is logically sound. That is, it is sound until the final line. The "contradiction" is not actually a contradiction...

4.10 Topology

Rather straightforward. Just assessing the students' understanding of the definition of a basis of a topology.

MATH 2001
QUIZ 10

(1) (1 pt) Write your name in the top right corner of the page.

(2) (6 pts) Recall that *a basis of a topology on A* is defined as follows.

Definition. Let A be a set, and let $\mathcal{B}$ be a set of subsets of A. The set $\mathcal{B}$ is a *basis for a topology on A* if the following properties are satisfied.

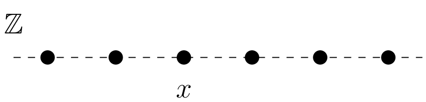
i For each $x \in A$, there exists a $B \in \mathcal{B}$ such that $x \in B$.

ii If $B_1, B_2 \in \mathcal{B}$, then for each $x \in B_1 \cap B_2$, there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

Complete the following exercises to determine if $\mathcal{B} = \{\{n, n+1, n+2\} : n \in \mathbb{Z}\}$ is a basis for a topology on $A = \mathbb{Z}$.

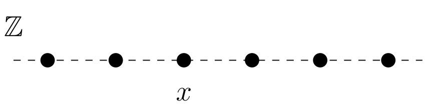
(a) Let $\mathcal{B} = \{\{n, n+1, n+2\} : n \in \mathbb{Z}\}$, and suppose $x \in \mathbb{Z}$.

Does there exist a $B \in \mathcal{B}$ such that $x \in B$? **If yes**, give a B that contains x **and** draw B in the picture below. **If not**, give a counterexample, or briefly explain why no such B exists.



Yes. Not drawn, but $\{x-2, x-1, x\}$, $\{x-1, x, x+1\}$, and $\{x, x+1, x+2\}$ are all valid basis elements.

(b) Suppose $B_1, B_2 \in \mathcal{B}$ and that $B_1 \cap B_2 \neq \varnothing$. For each $x \in B_1 \cap B_2$, does there exist a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$? **If yes**, give a brief explanation. **If not**, draw a B_1, B_2, and B_3 in the picture below that do not satisfy the condition.



No. For example if $B_1 = \{x-1, x, x+1\}$ and $B_2 = \{x, x+1, x+2\}$, then there does not exist a B_3 for which $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. More generally, any B_1 and B_2 where $|B_1 \cap B_2| < 3$ yields a counterexample.

(3) (3 pts) Let A be a set, let $\mathcal{B}$ be a basis for a topology on A, and let U be a subset of A. State what it means for U to be an *open set*.

The set U is open if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Date: May 5, 2016.

4.11 Relations

Again, straightforward. These last few quizzes are quite short because the end of the semester is nearing, and I wanted the class time to cover more material.

MATH 2001
QUIZ 11

(1) (1 pt) Write your name in the top right corner of the page.

(2) (3 pts each) Complete each of the following definitions. (If you don't like how my sentences start, feel free to write your own, but your statement must be complete (define all your variables etc.).

(a) Let A be a set, then R is a *relation* on A if

$R \subseteq A \times A$.

(b) Let R be a relation on A, then R is *transitive* if

whenever $(a, b), (b, c) \in R$, then $(a, c) \in R$.

(c) If R is an equivalence relation on A and $a \in A$, then the *equivalence class* of a is

$$[a] = \{b \in A : (a, b) \in R\}$$

(Set-builder notation is sufficient.)

(3) (3 pts each) Let $A = \{4, 9, 11\}$ and let R be the relation on A defined by

$$R = \{(a, b) : 3 \mid (a - b) \text{ or } 3 \mid (a + b)\}.$$

(a) Write out the relation R explicitly (list all of the elements of R in set notation).

$R = \{(4, 4), (9, 9), (11, 11), (4, 11), (11, 4)\}$

(b) Is R an equivalence relation? If **yes**, write out all of the equivalence classes of R. If **no**, give a brief explanation of why not (e.g. give a counter example).

Yes.

$$[4] = [11] = \{4, 11\}$$
$$[9] = \{9\}$$

Date: May 2, 2016.

4.12 Functions

MATH 2001
QUIZ 12

(1) (1 pt) Write your name in the top right corner of the page.

(2) (2 pts each) Complete each of the following definitions.

(a) A function $f\colon A \to B$ is *injective* if ...

$f(a) = f(b)$ implies $a = b$.

(b) A function $f\colon A \to B$ is *surjective* if ...

for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

(c) Let $f\colon A \to B$ be a function, and let $U \subseteq A$. Then

$$f(U) = \{f(x) \in B\colon x \in U\}$$

(d) Let $f\colon A \to B$ be a function, and let $V \subseteq B$. Then

$$f^{-1}(V) = \{x \in A\colon f(x) \in V\}$$

(3) (6 pts) Let $A = \{-1, 0, 1\}$, and consider the function $f\colon A^2 \to A$ defined by $f(x, y) = xy$.

(a) The function f **is** / **is not** (circle one) injective.

Is not.

(b) Justify your claim in part (a) by giving explicit values of the function. You don't have to write out the value of the function at every point—just give enough to confirm that the function is or isn't injective. (No need to write in full sentences either.)

$f(0, 0) = 0$ and $f(1, 0) = 0$ (but $(0, 0) \neq (1, 0)$).

(c) The function f **is** / **is not** (circle one) surjective.

Is.

(d) Justify your claim in part (c) by giving explicit values of the function.

$f(-1, 1) = -1$, $f(0, 0) = 0$, and $f(1, 1) = 1$.

(continued on reverse)

Date: May 4, 2016.

(4) (2 pts each) Let $A = \{-1, 0, 1\}$, and consider the function $f \colon A^2 \to \mathbb{Q}$ defined by $f(x, y) = x + y$. Write out each of the sets explicitly.

(a) $f^{-1}(\mathbb{N}) =$

$$\{(0, 1), (1, 0), (1, 1)\}$$

(b) $f(A^2) =$

$$\{-2, -1, 0, 1, 2\}$$

Chapter 5

Proof portfolio

Find everything in the `ProofPortfolio` folder.

These proofs were assigned over the course of the semester—approximately one new proof every two or three class periods. On every proof, I allowed rewrites. At the beginning of the semester, students were permitted two drafts before final copies were due, and in the latter half, one rewrite. I gave comments on all drafts, and often these comments were quite extensive. Generally, rewrites were due the class after I gave comments.

It goes without saying that the commenting process was extremely time consuming. Especially by the end of the semester, I was severely behind on returning comments in a timely manner. That being said, I would absolutely do this all again.

The only problem I would alter is Proof 12. It is just too much at once, and the first part of that problem is too messy.

I will not include samples of students' works here, so if you are interested in how I commented in documents, check out my "solutions" to quizzes 8 and 9.

Each student maintained their own tex file that contained all of these proofs, which allowed me to easily jump between assignments. The students maintained their own documents, and I expected them to keep their portfolios up to date as each new problem was posted. When assigning new problems, I posted the tex in a txt file on my webpage so that the students could easily copy the new problem into their Portfolio.

A condensed version of the Portfolio is attached below.

PROOF PORTFOLIO

AUTHOR

Contents

Preface

Keys to mathematical writing.

Date: May 9, 2016.

(1) **Guide the reader.** As with any piece of writing, it is important to provide the reader with context. The first line(s) of a proof should outline what will be proved and how the proof will be carried out. Longer proofs may have multiple interludes to remind the reader
   1. what you are trying to accomplish,
   2. what parts of the proof have been completed so far, and
   3. what part of the proof will be tackled next.

(2) **Write in complete English sentences.** Every statement should be a sentence. Sentences should be organized into paragraphs. The rules of spelling, punctuation, and grammar apply to mathematics as well.

(3) **Be precise.** Avoid using (read: do not use) ambiguous words and phrases. Wherever possible, be explicit about the objects to which you are referring.

(4) **Define all of your notation.** The first time you use a symbol, state explicitly what that symbol means (even if the symbol previously appeared in the statement of a problem, theorem, or definition).

(5) **Use appropriate symbols.** Mathematical symbols should match the context in which they are used. Mathematical phrases should integrate seamlessly into the surrounding text. It is a faux pas to begin a sentence with a mathematical symbol. *Be careful not to use the same symbol to represent multiple objects.*

(6) **Justify your claims.** For the most part, each sentence in the body of your proof should contain two statements:
   1. a statement of fact (usually) a logical consequence of the preceding statement, and
   2. justification for why the logical statement is true.

   Usually statements are justified by citing a definition or a theorem, or by providing simple algebraic steps. In some cases, more than one line of justification is needed.

(7) **Write for your peers.** Write arguments that can be understood by your classmates. Keep your statements simple, and strive for clarity. Use words that everyone can understand.

(8) **Write the complete statement of each definition.** Write out the precise statement of each definition you use in your proof. (You may want to do this before your proof, rather than within the proof, to avoid filling the proof with interjections. Writing out the definitions will help you learn the statements, and also provide a helpful reminder to the reader.

(9) **Proofread.** In fact, *read your proof out loud.* Do your statements read smoothly, or are there gaps? Do you find the need to insert words/phrases/pauses/interludes/etc. to make sense of or clarify your statements? Your readers should not have to guess to fill in blanks.

1. Prove that if $A = B$, then $A \subseteq B$ and $B \subseteq A$.

1.1. **First draft.** Due Monday, February 8 at 6:00 PM.

Proof. Write your proof here. □

1.2. **Second draft.**

Proof. After receiving my comments, write your revised proof here. (Do not make changes in the first section.) □

2. Prove that if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Remark. Combined with Proof 1, these two statements yield the following theorem.

Theorem 2.1 (Double containment)**.** *If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

The "if and only if" means that the statement can be read in either direction. Specifically, "$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$" is equivalent to the following statements:

(a) if $A = B$, then $A \subseteq B$ and $B \subseteq A$, and
(b) if $A \subseteq B$ and $B \subseteq A$, then $A = B$.

This result (in particular, statement (b)) is one of the most common methods for showing that two sets are equal. Since this result relies on showing that each of the two sets is contained within the other, we call this result the Double Containment (or Double Inclusion) Principle.

2.1. **First draft.** Due Wednesday, February 10 at 6:00 PM.

Proof. Write your proof here. □

3. Prove that if $C \subseteq A \cap B$, then $C \subseteq A$ and $C \subseteq B$.

3.1. **First draft.** Due Monday, February 15 at 6:00 PM.

Proof. □

4. Prove that $\{x \in \mathbb{Z} : 55 \mid x\} \subseteq \{x \in \mathbb{Z} : 11 \mid x\}$.

4.1. **First draft.** Due Friday, February 19 at 6:00 PM.

Proof. □

5. Prove that $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

5.1. **First draft.** Due Wednesday, February 24 at 6:00 PM.

Proof. □

6. Prove that $\{6n + 9 : n \in \mathbb{Z}\} = \{6n - 3 : n \in \mathbb{Z}\}$.

Remark. To prove that two sets are equal, give a double containment argument.

6.1. **First draft.** Due Monday, February 29 at 6:00 PM.

Proof. □

7. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Remark. To prove that two sets are equal, give a double containment argument.

7.1. **First draft.** Due Friday, March 4 at 6:00 PM.

Proof. □

8. Prove that $\displaystyle\bigcap_{y>0,\, y\in\mathbb{R}} A_y = \underline{\hspace{4cm}}$,

where $A_y = (-y, y) \subseteq \mathbb{R}$ is an open interval. It is up to you to determine the set on the right and to prove the equality. If possible, determine the set on the right explicitly.

8.1. **First draft.** Due Wednesday, March 9 at 6:00 PM.

Proof. □

9. Suppose x, y, and $z \in \mathbb{Z}$, and $x \neq 0$. Prove that if $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

Remark. Give a proof by contrapositive.

9.1. **First draft.** Due Monday, March 14 at 6:00 PM.

Proof. □

10. Prove that $\sqrt{6}$ is irrational.

Remark. Give a proof by contradiction.

10.1. **First draft.** Due Wednesday, March 16 at 6:00 PM.

Proof. □

11. Prove that $\mathcal{B} = \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$.

11.1. **First draft.** Due Wednesday, March 30 at 6:00 PM.

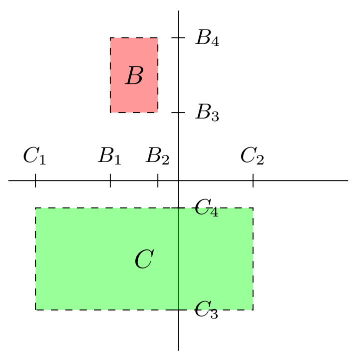
Proof. □

12. Determine if these sets are open and/or closed.

Let $\mathcal{B}$ be the set of "open rectangles" in $\mathbb{R}$:

$$\mathcal{B} = \{(a, b) \times (c, d) \subseteq \mathbb{R}^2 : (a, b) \subseteq \mathbb{R}, (c, d) \subseteq \mathbb{R}\}.$$

Here are two examples of elements in $\mathcal{B}$: $B = (B_1, B_2) \times (B_3, B_4)$ and $C = (C_1, C_2) \times (C_3, C_4)$.



The set $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$. (You do not have to prove that $\mathcal{B}$ is a basis.)

Consider the following sets.

(1) The interior of the unit circle: $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
(2) The integer lattice $\mathbb{Z}^2$: $U = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

For each set U, complete the following.

a. Determine if U is open or not open. Prove your claim.
b. Determine if U is closed or not closed. Prove your claim.

In the end, you should have four short proofs.

12.1. **First draft.** Due Monday, April 4 at 6:00 PM.

(1) The interior of the unit circle: $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.
  (a) Claim: U is (pick one: open/not open).
  Proof. $\square$
  (b) Claim: U is (pick one: closed/not closed).
  Proof. $\square$
(2) The integer lattice $\mathbb{Z}^2$: $U = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathbb{Z}\}$.
  (a) Claim: U is (pick one: open/not open).
  Proof. $\square$
  (b) Claim: U is (pick one: closed/not closed).
  Proof. $\square$

13. Prove that the intersection of two open sets is open.

13.1. **First draft.** Due Friday, April 8 at 6:00 PM.

Proof. $\square$

14. Prove that R is an equivalence relation on $\mathbb{Z}$.

Let $A = \mathbb{Z}^2 - \{(0, 0)\}$, and let R be the relation on A defined by

$$R = \{((a, b), (c, d)) : ad - bc = 0\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.

14.1. **First draft.** Due Friday, April 8 at 6:00 PM.

Proof. $\square$

15. Prove the following correspondence between equivalence relations and partitions.

Let A be a set.

(1) Prove that if R is an equivalence relation on A, then $\{[a] : a \in A\}$ is a partition of A.
(2) Prove that if P is a partition of A, then $R = \{(x, y) : x, y \in X \text{ and } X \in P\}$ is an equivalence relation on R.

15.1. **First draft.** Due Wednesday, April 20 at 6:00 PM.

Proof. $\square$

Chapter 6

Final exam(s)

Final exams are located in the `Quizzes` folder.

I wrote two final exams. Admittedly, there is not a good balance between the two tests. Not every problem on these tests made the final cut.

MATH 2001
FINAL EXAM

(1) (a) Complete each of the following definitions.

(i) Let A be a set. The *power set* of A is

(ii) Let A and B be sets. The *difference*, $A - B$, is

(b) Consider the following proof.

Proof. Suppose $X \in \mathscr{P}(A-B)$. Then $X \in A-B$, which implies that if $y \in X$, then $y \in A-B$. Since $y \in X$ and $y \in A$, then $X \in A$, and since $y \in X$ and $y \notin B$, then $X \notin B$. Therefore $X \in \mathscr{P}(A)$ and $X \notin \mathscr{P}(B)$, so $X \in \mathscr{P}(A) - \mathscr{P}(B)$. $\square$

(i) What claim is the author attempting to prove in this argument?

(ii) There are a number of instances in the proof where $\in$ (or $\notin$) inappropriately. Circle each of of those instances.

(iii) Outside of the notational errors, there is one flaw in this argument that renders the argument incorrect. What is that flaw?

Date: May 5, 2016.

(c) State the definition of the *union* of two sets.

(d) Sketch a proof that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

(2) (a) State each of the following definitions.

(i) Let f be a relation from A to B. The relation f is a *function from A to B* if

(ii) A function $f\colon A \to B$ is *injective* if

(b) Let $f\colon A \to A$ be an injective function. Prove that f^2 is an injective function, where $f^2 = f \circ f$ is the composition of f with itself.

(c) Prove that f^n is injective. (Again, $f^n = f \circ f \circ \cdots \circ f$ is the composition of f with itself n times.)

(3) Recall the following definitions.

Definition. Let A be a set, and let $\mathcal{B} \subseteq \mathscr{P}(A)$. The set $\mathcal{B}$ is a *basis for a topology on* A if each of the following is satisfied.

(a) For each $x \in A$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(b) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

Definition. Let A be a set and let $P \subseteq \mathscr{P}(A)$. The set P is a *partition of* A if each of the following is satisfied.

(a) $\varnothing \notin P$.

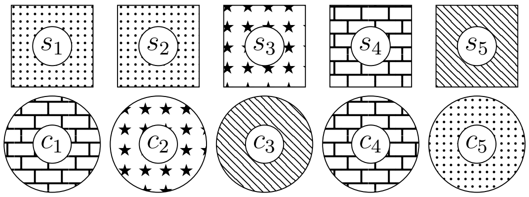
(b) $\bigcup_{X \in P} X = A$.

(c) If $X_1, X_2 \in P$, then $X_1 \cap X_2 = \varnothing$ if and only if $X_1 \neq X_2$.

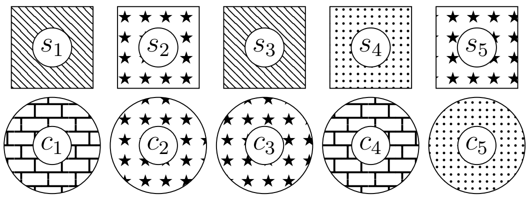
Let A be a set and let P be a partition of A. Prove that P is a basis for a topology on A.

(4) Consider the following diagrams

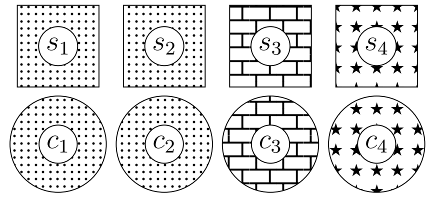
I.



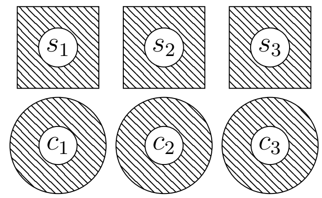
II.



III.



IV.



Let $P(c_i)$ and $P(s_j)$ denote the pattern on the i-th circle and j-th square, respectively.

(a) Write the contrapositive, negation, and converse of the statement below.

(b) In each box, write the roman numeral of the diagram that satisfies the adjacent statement.

$\star$ If there exists an i for which $P(c_i)$ is "lines", then there exists a j for which $P(s_j)$ is "lines".

Contrapositive:

Negation:

Converse:

MATH 2001
FINAL EXAM

(1) (1 pt) Write yer name at the top wherever you please.

(2) (a) (2 pts) According to de Morgan's laws, if A and B are sets, then

$$\overline{A \cup B} =$$

$$\overline{A \cap B} =$$

(Just complete the statements; no proofs necessary.)

(b) (4 pts) Complete the following definitions given that A is a set and $\mathcal{B}$ is a basis for a topology on A.

(i) If $U \subseteq A$, then U is *open* if

(ii) If $V \subseteq A$, then V is *closed* if

(c) (10 pts) Give a proof by induction that the union of finitely many closed sets is closed. That is, prove that $\bigcup_{i=1}^{n} A_i$ is closed given that each A_i is a closed set.

[Hints: for the base case, prove that the union of two closed sets is closed. It may be helpful to recall that in Proof 13, you proved that the intersection of two open sets is open.]

Date: May 7, 2016.

(3) (a) Consider the following definition. (This is not a definition that we have seen in the class thus far.)

Definition. Let $U \subseteq \mathbb{R}$. The number m is the *maximum of* U if

- $m \in U$, and for all $x \in U$, we have $x \leq m$.

(i) (5 pts) Suppose $m \in \mathbb{R}$. What does it mean if m is *not* the maximum of U? (In other words, what is the negation of the bulleted statement?)

The number m is not the maximum of U if ...

(ii) (10 pts) Does $\{\frac{-1}{10^n} : n \in \mathbb{N}\}$ have a maximum? Prove your claim.

(b) (10 pts) Consider the following theorem and definition.

Theorem. *If $a, b \in \mathbb{Q}$, then there exists $c \in \mathbb{Q}$ such that $a < c < b$.*

Definition. A set $C \subset \mathbb{R}$ is a *cut* if it satisfies each of the following properties:

- C is a non-empty, proper subset of $\mathbb{Q}$, meaning: $C \subset \mathbb{Q}$, $C \neq \mathbb{Q}$, and $C \neq \emptyset$.
- For any two rational numbers $a, b \in \mathbb{Q}$, if $a > b$ and $a \in C$, then $b \in C$.
- C has no maximal element.

Prove prove that if $r \in \mathbb{Q}$ and $C_r = \{q \in \mathbb{Q} : q < r\}$, then C_r is a cut.

(4) (12 pts) Complete each of the following definitions.

(a) If A and B are sets, then the *Cartesian product*, $A \times B$, is

(b) If A is a set, then R is a *relation* on A if

(c) If R is a relation on A, then R is an *equivalence relation* if

(d) If R is an equivalence relation on A and $x \in A$, then the *equivalence class* $[x]$ is

(e) If f is a relation from A to B, then f is a *function* if

(f) If A is a set, then the *power set of* A is

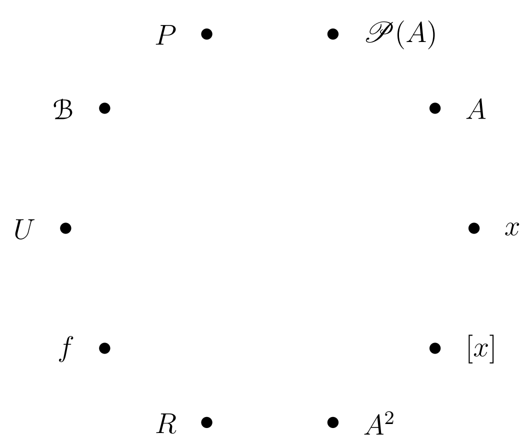
(5) (10 pts)

- Let A be a set.
- Let P be a partition of A.
- Let R be an equivalence relation on A.
- Let x be an element of A.
- Let $[x]$ be the equivalence class of x (defined by the relation R).
- Let f be a function from A to A.
- Let $\mathscr{P}(A)$ be the power set of A.
- Let $\mathcal{B}$ be a basis for a topology on A.
- Let U be an open subset of A (with respect to the basis $\mathcal{B}$).
- Let A^2 be the Cartesian product of A with itself.

In the diagram below,

- draw a **single-headed** arrow from one item to another if the first is an **element** of the other;
- draw a **double-headed** arrow from one item to another if the first is a **subset** of the other.

Points: 1 point per correct arrow, -0.5 points per incorrect arrow. It is possible to earn more than 10 points on this problem. It is also possible to earn fewer than 0 points on this problem.



To keep things tidy, you might want to work this one out on scrap paper before drawing your final answer on this page.

(6) (10 pts) True or false.

T **F** : If R is an equivalence relation on A, then R is a function on A.

T **F** : If R is an equivalence relation on A, then $R = R^{-1}$.

T **F** : If A is an open set and $U \subseteq A$, then U is open.

T **F** : If $f\colon A \to B$ is a function and $x \in A$, then $f(x) \in B$.

T **F** : If $f\colon A \to B$ is a function and $x \in B$, then $f^{-1}(x) \in A$.

Let $f\colon A \to B$ be a function. Given sets U and V where $U \subseteq A$ and $V \subseteq B$, let

$$f_{U,V} = f \cap (U \times V).$$

T **F** : $f_{U,B}\colon U \to B$ is a function for any $U \subseteq A$.

T **F** : There exists a $U \subseteq A$ such that $f_{U,B}^{-1}\colon B \to U$ is a function.

T **F** : $f_{A,V}\colon A \to V$ is a function for any $V \subseteq B$.

T **F** : There exists a $V \subseteq B$ such that $f_{A,V}^{-1}\colon V \to A$ is a function.

T **F** : $f_{U,V}\colon U \to V$ is a function for any $U \subseteq A$ and $V \subseteq B$.

(1 pt) What is my age?

Odbayar graciously awards you ...

Appendix A

Flashcards

Here are flashcards with the vocabulary relevant to my course. The cards are formatted as to be printed out on 2" $\times$ 3.5" business card stock. Perforated card stock is generally cheap and durable, so this template makes for a very fast and easy way for students to make their own set of cards. Of course, you can also print on regular paper and cut them out.

The flashcard template is very easy to use. I strongly recommend it to everyone.

There seems to be a minor issue with the margins in the flashcard package. I do not know if that it an oversight on the part of the developer, or if there is a conflict with one of the packages I am using. In any case, I needed to adjust the top margin of the page (`\geometry{top=.35in}`) so that the borders of the flashcards match the perforations on business card stock.

Definition

Set and element

Sets

Definition

Cardinality of a set

Sets

Definition

Empty set (null set)

Sets

Definition

Finite and infinite sets

Sets

Definition

Ordered pair / n-tuple

Sets

Definition

Cartesian product

Sets

Definition

Subset

Sets

Definition

Proper subset

Sets

Definition

Power set

Sets

Definition

Partition

Sets

The *cardinality* of a set is the number of elements in the set.
The cardinality of the set A is denoted by $|A|$ or $\#A$.

A *set* is a collection of objects; the objects in the set are called *elements*.
If A is a set, and a is an element of A, then we write $a \in A$.

A set is *finite* if its cardinality is finite, that is, the set contains finitely many elements.
A set is *infinite* if it contains infinitely many elements.

The *empty set* is the set that contains zero elements.
The empty set is denoted by $\varnothing$.

If A and B are sets, the *cartesian product* $A \times B$ is the set of ordered pairs where the first element is from A, and the second is from B.

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

$$(a, b) \in A \times B \qquad \Leftrightarrow \qquad a \in A \text{ and } b \in B$$

An *ordered pair* is an ordered list of two elements.
More generally, an *ordered n-tuple* is an ordered list of n elements. The standard notation is to use a comma separated list enclosed by parenthesis: $(a_1, a_2, \ldots, a_n)$.

The set B is a *proper* subset of A if B is a subset of A that is not equal A, and we write $B \subset A$.

$$B \subset A \qquad \Leftrightarrow \qquad B \subseteq A \text{ and } B \neq A$$

A set A is a *subset* of a set B if every element of A is an element of B.

$$A \subseteq B \qquad \Leftrightarrow \qquad x \in A \Rightarrow x \in B$$

Let A be a set, and let $P \subseteq \mathscr{P}(A)$. The set P is a *partition of* A if

1. $\bigcup_{X \in P} X = A$;
2. if $X_1, X_2 \in P$, then $X_1 \cap X_2 = \varnothing \Leftrightarrow X_1 \neq X_2$.

The *power set* of a set A is the set of all subsets of A.
The power set of A is denoted by $\mathscr{P}(A)$.

$$\mathscr{P}(A) = \{B : B \subseteq A\}$$

Definition

Set equality

Sets

Definition

Union

Sets

Definition

Finite and infinite union

Sets

Definition

Intersection

Sets

Definition

Finite and infinite intersection

Sets

Definition

Set difference

Sets

Definition

Disjoint sets

Sets

Theorem

Double containment principle

Sets

Definition

Complement

Sets

Theorem

De Morgan's laws

Sets

The *union* of two sets, A and B, is the set of all element in A or in B.
The union of these sets is denoted by $A \cup B$.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$x \in A \cup B \quad \Leftrightarrow \quad x \in A$ or $x \in B$

Two sets, A and B, are *equal* if all the elements of A are elements of B and vice versa.

$A = B \quad \Leftrightarrow \quad x \in A$ if and only if $x \in B$

The *intersection* of two sets, A and B, is the set of all element in A and in B.
The intersection of these sets is denoted by $A \cap B$.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

$x \in A \cap B \quad \Leftrightarrow \quad x \in A$ and $x \in B$

A *finite union* is the union of finitely many sets. An *infinite union* is the union of infinitely many sets.
Let $A_1, A_2, A_3, \ldots$ be sets, then

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } 1 \leq i \leq n\}$$
$$\bigcup_{i \in \mathbb{N}} A_i = \{x : x \in A_i \text{ for some } i \in \mathbb{N}\}$$

If A and B are sets, the, is the difference $A - B$ is the set of elements in A that are not in B.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

$x \in A - B \quad \Leftrightarrow \quad x \in A$ and $x \notin B$

A *finite intersection* is the intersection of finitely many sets. An *infinite intersection* is the intersection of infinitely many sets.
Let $A_1, A_2, A_3, \ldots$ be sets, then

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for all } 1 \leq i \leq n\}$$
$$\bigcap_{i \in \mathbb{N}} A_i = \{x : x \in A_i \text{ for all } i \in \mathbb{N}\}$$

Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

$A = B \quad \Leftrightarrow \quad A \subseteq B$ and $B \subseteq A$

Two sets, A and B, are *disjoint* if $A \cap B = \varnothing$.

For any sets A and B,

$$(A \cap B)^c = A^c \cup B^c,$$
$$(A \cup B)^c = A^c \cap B^c.$$

The *complement* of a set A is the set of all elements that are not in A, and is denoted by A^c or $\overline{A}$.
If $A \subseteq B$, then the *complement of* A *in* B is the set of elements in B that are not in A, i.e. $A^c = B - A$.

$$A^c = \{x : x \notin A\}$$

$x \in A^c \quad \Leftrightarrow \quad x \notin A$

Definition

Statement

Logic

Definition

Logical statement

Logic

Definition

Logical and

Logic

Definition

Logical or

Logic

Definition

Logical negation

Logic

Definition

Implies

Logic

Definition

Converse

Logic

Definition

If and only if

Logic

Definition

Contrapositive

Logic

Definition

Universal quantifier: for all

Logic

A *statement* is a sentence or mathematical expression that is definitely true or definitely false.

A *statement* is a sentence or mathematical expression that is definitively true or definitively false.

The statement "P *or* Q" is true if P is true or Q is true (or both statements are true). The statement "P and Q" is false only if both P is false and Q is false.

$P \vee Q$ is true	$\Leftrightarrow$	P is true or Q is true

The statement "P *and* Q" is true if both P is true and Q is true. Otherwise "P and Q" is false.

$P \wedge Q$ is true	$\Leftrightarrow$	P is true and Q is true

The statement "P *implies* Q" ($P \Rightarrow Q$) is false if P is true and Q is false. Otherwise the statement is true.

$P \Rightarrow Q$ is false	$\Leftrightarrow$	P is false and Q is true

The *negation* of a statement P is the statement $\neg P$. The statement $\neg P$ is true if P is true. The statement of $\neg P$ is false if P is true.

P is true (resp. false)	$\Leftrightarrow$	$\neg P$ is false (resp. true)

The statement "P *if and only if* Q" ($P \Leftrightarrow Q$) is equivalent to the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. In other words, $P \Leftrightarrow Q$ is true if both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true.

$P \Leftrightarrow Q$	$\Leftrightarrow$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

The *converse* of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. In general, these two statements are independent, meaning that the truthfulness of one statement does not determine the truthfulness of the other.

The *for all/each/every/any* statement takes the form: "for all P, we have Q." In other words, Q is true whenever P is true. In this light, "for all" statements can often be reworded as "if-then" statements (and vice versa).

$\forall P$, we have Q	$\Leftrightarrow$	$P \Rightarrow Q$

The *contrapositive* of the statement "if P, then Q" is the statement "if $\neg Q$, then $\neg P$". These statements are equivalent, meaning that they are either both true or both false.

$P \Rightarrow Q$	$\Leftrightarrow$	$\neg Q \Rightarrow \neg P$

Definition

Existential quantifier: there exists

Logic

Negation of $P \wedge Q$

Logic

Negation of $P \vee Q$

Logic

Negation of $P \Rightarrow Q$

Logic

Negation of $\forall P,$ we have Q

Logic

Negation of $\exists P$ such that Q

Logic

Definition

Theorem

Logic

Definition

Proof

Logic

Definition

Definition

Logic

Definition

List and entries

Counting

$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

The *there exists* statement takes the form: "there exists P such that Q." This statement is true if there is at least one case where P is true and Q is true. (It maybe that there are many cases where P is false but Q is true.)

$\exists P$ such that Q	$\Leftrightarrow$	it is sometimes the case that $P \Rightarrow Q$

$$\neg(P \Rightarrow Q) = P \wedge \neg Q$$

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$

$$\neg(\exists P, \text{ such that } Q) = \forall P \text{ we have } \neg Q$$

$$\neg(\forall P, \text{ we have } Q) = \exists P, \text{ such that } \neg Q$$

A *proof* of a theorem is a written verification that shows that the theorem is definitely and unequivocally true.

A *theorem* is a mathematical statement that is true and can be (and has been) verified as true.

A *list* is an ordered sequence of objects. The objects in the list are called *entries*. Unlike sets, the order of entries matters, and entries may be repeated.

A *definition* is an exact, unambiguous explanation of the meaning of a mathematical word or phrase.

Definition

List length

Counting

Definition

Empty list

Counting

Definition

Factorial

Counting

Theorem

Binomial theorem

Counting

Theorem

Addition principle

Counting

Definition

List equality

Counting

Theorem

Multiplication principle

Counting

Definition

n **choose** k

Counting

Theorem

Inclusion-exclusion

Counting

Definition

Even

Counting

Two lists L and M are *equal* if they have the same length, and the i-th entry of L is the i-th entry of M.

The *length* of a list is the number of entries in the list.

Suppose in making a list of length n there are a_i possible choices for the i-th entry. Then the total number of different lists that can be made in this way is $a_1 a_2 a_3 \cdots a_n$.

The *empty list* is the list with no entries, and is denoted by $()$.

If n and k are integers, and $0 \le k \le n$, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If n is a non-negative integer, then $n!$ is the number of non-repetitive lists of length n that can be made from n symbols. Thus $0! = 1$, and if $n > 1$, then $n!$ is the product of all integers from 1 to n. That is, if $n > 1$, then $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

If A and B are sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.

If n is a non-negative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

An integer a is *even* if there exists an integer b such that $a = 2b$.

a is even	$\Leftrightarrow$	$a = 2b$ for some $b \in \mathbb{Z}$

If $A_1, A_2, \ldots, A_n$ are disjoint sets, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.

Definition

Odd

Counting

Definition

Parity

Counting

Definition

Composite

Counting

Definition

Least common multiple

Counting

Theorem

Division algorithm

Counting

Definition

Divides

Counting

Definition

Prime

Counting

Definition

Greatest common divisor

Counting

Theorem

Well-ordering principle

Counting

Definition

Basis for a topology

Topology

If a and b are integers, then b *divides* a if there exists an integer q such that $a = qb$. In this case, b is a *divisor* of a, and a is a *multiple* of b.

$b \mid a$	$\Leftrightarrow$	$a = qb$ for some $q \in \mathbb{Z}$

An integer a is *odd* if there exists an integer b such that $a = 2b + 1$.

a is odd	$\Leftrightarrow$	$a = 2b + 1$ for some $b \in \mathbb{Z}$

A positive integer $p > 1$ is *prime* if the only divisors of p are 1 and p.

$p > 1$ is *prime*	$\Leftrightarrow$	a has exactly two positive divisors: 1 and p

Two integers have the *same parity* if they are both even or both odd. Otherwise they have *opposite parity.*

The *greatest common divisor* of two integers a and b, denoted $\gcd(a, b)$, is the largest integer that divides both a and b.

A positive integer a is *composite* if there exists a positive integer $b > 1$ satisfying $b \mid a$.

$a > 1$ is composite	$\Leftrightarrow$	$b \mid a$ and $1 < b < a$

Every non-empty subset of $\mathbb{N}$ contains a least element.

The *least common multiple* of two integers a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer is a multiple of both a and b.

Let A be a set, and let $\mathcal{B} \subseteq \mathscr{P}(A)$. The set $\mathcal{B}$ is a *basis for a topology on* A if the following are satisfied.

1. If $x \in A$, then there exists $B \in \mathcal{B}$ such that $x \in B$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

Given integers a and b with $b > 0$, there exist unique integers q and r that satisfy $a = bq + r$, where $0 \leq r < b$.

Definition

Open set

Topology

Definition

Closed set

Topology

Definition

Relation on a set

Relations

Definition

Reflexive

Relations

Definition

Symmetric

Relations

Definition

Transitive

Relations

Definition

Equivalence relation

Relations

Definition

Equivalence class

Relations

Definition

Relation between sets

Relations

Definition

Inverse relation

Relations

Let A be a set, let $\mathcal{B}$ be a basis for a topology on A, and let $U \subseteq A$. The set U is *closed* if U^c is open.

Let A be a set, let $\mathcal{B}$ be a basis for a topology on A, and let $U \subseteq A$. The set U is *open* if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$.

Let R be a relation on A. The relation R is *reflexive* if $a \in A$ implies that $(a, a) \in R$.

Let A be a set. The set R is a *relation on A* if $R \subseteq A^2$.

Let R be a relation on A. The relation R is *transitive* if $(a, b), (b, c) \in R$ implies that $(a, c) \in R$.

Let R be a relation on A. The relation R is *symmetric* if $(a, b) \in R$ implies that $(b, a) \in R$.

Let R be an equivalence relation on A, and let $a \in A$. The *equivalence class of a* is the set

$$[a] = \{b \in A : (a, b) \in R\}.$$

$$x \in [a] \qquad \Leftrightarrow \qquad (a, x) \in R$$

Let R be a relation on A. The relation R is an *equivalence relation* (on A) if it is reflexive, symmetric, and transitive.

Let R be a relation from A to B. The *inverse* of R is the relation from B to A given by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

$$(x, y) \in R^{-1} \qquad \Leftrightarrow \qquad (y, x) \in R$$

Let A and B be sets. The set R is a *relation from A to B* if $R \subseteq A \times B$.

Definition

Function

Relations

Definition

Domain/codomain/image

Relations

Definition

Image of a set

Relations

Definition

Inverse image of a set

Relations

Let f be a function from A to B. The *domain* of f is A. The *codomain* of f is B, and the *image* of f is the set $\{b \in B : (a, b) \in f\}$. In other words, the image of f is the set $\{f(a) : a \in A\}$.

Let R be a relation from A to B. The relation R is a *function* if for each $a \in A$, R contains a unique element of the form (a, b). In this case, we write $R(a) = b$.

Let f be a function from A to B, and let $V \subseteq B$. Then the *inverse image of* V (or *preimage of* V) is the set

$$f^{-1}(V) = \{x \in A : f(x) \in V\}.$$

$x \in f^{-1}(V)$	$\Rightarrow$	$f(x) \in V$
$y \in V$	$\Rightarrow$	$y = f(x)$ for some $x \in f^{-1}(V)$

Let f be a function from A to B, and let $U \subseteq A$. Then the *image of* U is the set

$$f(U) = \{f(x) \in B : x \in U\}.$$

$y \in f(U)$	$\Rightarrow$	$y = f(x)$ for some $x \in U$
$x \in U$	$\Rightarrow$	$f(x) \in f(U)$

Appendix B

LaTeX Guide

All files related to this guide are found in the `LaTeX_Guide` folder.

I wrote this as a quick, searchable reference for any tex that would be relevant to this course (and quite a bit more). Nearly every item is accompanied by an example that includes what the inputs and outputs should look like. Moreover, many expressions come with their English translation so that students can see how each expression is read.

There are a few commands that are specific to the header of my tex files. Generally, I keep `TexTemplate.tex` as the most up to date header. The `TexTemplate.tex` file also serves as a template for book exercises.

If you want a stand-alone version of this document, check out `LaTeX_Guide_Title.pdf`. The LaTeX Guide itself has a chapter heading because I originally wrote it as an appendix for a textbook.

Occasionally I think to add more items to this document. Feel free to send me a message if you think something should be changed or added.

Chapter 1

An introduction to LaTeX

1.1 Resources

In this course, all written work must be done in LaTeX. Software for composing LaTeX documents is freely available online, and there are also a number of free-to-use websites for creating collaborative LaTeX documents (useful for group assignments).

Links:

1. LaTeX wikibook: All things LaTeX.

   `http://en.wikibooks.org/wiki/LaTeX`

2. ShareLaTeX: Collaborative `.tex` documents.

   `https://www.sharelatex.com`

3. Overleaf: Another website for collaborative .tex-ing.

   `https://www.overleaf.com`

4. Detexify: Looking for a symbol but you do not know what it is called? Look up a symbol by drawing it! (Tends to be hit-or-miss, but still generally useful.)

   `http://detexify.kirelabs.org/classify.html`

5. Symbols list: More than you will ever need ever.

   `http://www.tex.ac.uk/tex-archive/info/symbols/comprehensive/symbols-a4.pdf`

6. TikZ: You might learn how to draw pictures one day...

   `http://www.texample.net/tikz/examples`

7. Acquiring LaTeX: If you are interested in obtaining LaTeX for personal use...

   `http://latex-project.org/ftp.html`

8. Writing with good practices: A reference.

   `http://www.math.illinois.edu/~dwest/grammar.html`

1.2 Modes

There are two primary *modes* in which LaTeX is written: *text mode*, and *math mode*. Text mode is used for text; write as you would in any word processor. Anything mathematical should be written in math mode. A dollar sign (`$`) is used to transition between the two modes.

Input:

```
A function $f \colon A \to \bbr$ is \emph{continuous} if for each
$\epsilon > 0$, there exists a $\delta > 0$, such that if $|x-c| < \delta$,
then $|f(x) - f(c)| < \epsilon$.
```

Output:

A function $f \colon A \to \mathbb{R}$ is *continuous* if for each $\epsilon > 0$, there exists a $\delta > 0$, such that if $|x-c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Math mode has two styles: math can be written *in-line* (as in the example above using dollar signs) or it sectioned away from text and be *displayed*. Some symbols will be type-set differently depending on the style. You can force displayed math to appear in-line using the command `\displaystyle` (or `\ds`†) in math mode. However, if you are going to write display-style math, you might as well place it in the `align` *environment* (see the next section).

Input:

```
In-line math: $\lim_{x \to \infty} \int_1^x \frac{1}{x}\,dx$,
and forced displaystyle in-line: $\ds\lim_{x \to \infty} \int_1^x
\frac{1}{x}\,dx$.
```

Output:

In-line math: $\lim_{x \to \infty} \int_1^x \frac{1}{x^2}\,dx$, and forced displaystyle in-line: $\displaystyle\lim_{x \to \infty} \int_1^x \frac{1}{x^2}\,dx$.

1.3 Environments

In short, an *environment* is simply a set of formatting rules that affects how text and math are displayed, among other things. Every environment is evoked by the command `\begin{[environment]}` and ended by `\end{[environment]}`. Everything between these commands is subject to the rules of that environment.

1.3.1 Document

Every `.tex` file must contain a `document` environment. The TeX compiler will only output what you write in this environment.

```
\begin{document}
[Turn this into a pdf please!]
\end{document}
```

1.3.2 Homework, definitions, theorems, and proof

The important statements in mathematics each have their own environment; these include `defn`†, `thm`†, `prop`†, `lem`†, and `conj`†. These key words are intrinsic to my `.tex` file; if you use someone else's file, the key words used to call these environments may not be the same.

Definition 1.3.1. A precise and unambiguous statement that gives meaning to a *key word*. Use `\emph{[key word]}` to highlight the key word.

Definitions *do not require proof.*

```
\begin{defn} ... \end{defn}
```

Theorem 1.3.2. *Theorems are major mathematical results. Theorems, along with Propositions and Lemmas, are claims that require* proof*. Unproved claims are called* conjectures*.*

```
\begin{thm} ... \end{thm}
```

Proposition 1.3.3. *Propositions are mathematical results, but they generally do not carry the same weight as a theorem. Propositions require proof.*

```
\begin{prop} ... \end{prop}
```

Lemma 1.3.4. *Lemmas are small, often technical, results. Generally, lemmas are 'helpful facts' that are needed to prove much larger results. Lemmas require proof.*

Claim 1.3.5. A catch-all term. Claims require proof.

```
\begin{claim} ... \end{claim}
```

Proof. A *proof* is an irrefutable, deductive argument. Be aware that there is a significant difference between giving evidence in support of a claim and proving a claim; examples do not constitute a proof. Proofs should be written in the `proof` environment. □

```
\begin{proof} ... \end{proof}
```

ANSWER: For homework problems that are not proofs, use the `answer`† environment. □

```
\begin{answer} ... \end{answer}
```

Conjecture 1.3.6. A conjecture is an unproven statement.

```
\begin{conj} ... \end{conj}
```

Naming definitions, theorems, etc.

You can add a 'name' to a definition, theorem, etc. by placing the name in square brackets immediately following the `\begin{[environment]}` command. A few examples:

```
\begin{defn}[even]
An integer $a$ is \emph{even} if $a = 2b$ for some integer $b$.
\end{defn}
```

Definition 1.3.7 (even)**.** An integer a is *even* if $a = 2b$ for some integer b.

```
\begin{thm}[Fermat's little theorem]
If $a$ is an integer and $p$ is a prime, then $a^p \equiv a \pmod p$.
\end{thm}
```

Theorem 1.3.8 (Fermat's little theorem)**.** *If a is an integer and p is a prime, then $a^p \equiv a \pmod p$.*

Suppressing numbers

The automated numbering of definitions, theorems, etc. can be suppressed with an asterisk (`*`).

```
\begin{defn*}
This definition has no number.
\end{defn*}
```

Definition. This definition has no number.

```
\begin{lem*}[name]
This lemma has a ‘name’, but no number.
\end{lem*}
```

Lemma (name)**.** *This lemma has a ‘name’, but no number.*

1.3.3 Lists

There are a number of environments that support lists, but I will only discuss `enumerate`.

1. Each numbered item in the list is specified by the key word `\item`.
2. Numbering is automated...

$\sqrt{5}$. ...but you can customize individual items using `\item[...]`.

3. Lists can be nested by calling `enumerate` again.
   (a) Again, numbering is automated.
   (b) And you can continue to nest lists.
      i. These lists may be customized further, but this should be sufficient for now.

```
\begin{enumerate}
\item Each numbered item in the list is specified by the key word \cverb;\item;.
\item Numbering is automated...
\item[$\sqrt 5$.]  ...but you can customize individual items using \cverb;\item[...];.
\item Lists can be nested by calling \everb;enumerate; again.
\begin{enumerate}
\item Again, numbering is automated.
\item And you can continue to nest lists.
\begin{enumerate}
\item These lists may be customized further, but this should be sufficient for now.
\end{enumerate}
\end{enumerate}
\end{enumerate}
```

1.3.4 Displayed equations

There are a number of environments that produce displayed math. My preferred environments are `align` and `align*`. Every line in `align` will be numbered; lines in `align*` will not.

Input:

```
\begin{align}
\sum_{n=1}^\infty \frac{1}{n^2} = \prod_{p \text{ prime}} \frac{1}{1-p^{-2}}.
\end{align}
```

Output:

$$\sum_{n=1}^\infty \frac{1}{n^2} = \prod_{p \text{ prime}} \frac{1}{1-p^{-2}}. \tag{1.1}$$

The equation is numbered (to the right of the equation) so that it may be referenced elsewhere in the document.

The `align` environment allows for multiple aligned columns across lines. Columns are separated using `&` and new lines are designated by `\\`. Columns alternate between right and left justification.

Input:

```
\begin{align*}
    e^{i\pi} &= \sum_{n=1}^\infty\frac{(i\pi)^n}{n!}
        && \text{(Taylor series for $e^x$)} \\
    &= \sum_{n=1}^\infty \frac{\pi^{2n}}{(2n)!}
        + i\sum_{n=1}^\infty\frac{\pi^{2n+1}}{(2n+1)!}
        && \text{(rearrangement of terms)} \\
    &= \cos(\pi) + i \sin(\pi)
        && \text{(Taylor series for $\cos(x)$ and $\sin(x)$)} \\
    &= -1. && \text{(simplifying the previous expression)}
\end{align*}
```

Output:

$$\begin{aligned}
e^{i\pi} &= \sum_{n=1}^\infty\frac{(i\pi)^n}{n!} && \text{(Taylor series for e^x)} \\
&= \sum_{n=1}^\infty \frac{\pi^{2n}}{(2n)!} + i\sum_{n=1}^\infty\frac{\pi^{2n+1}}{(2n+1)!} && \text{(rearrangement of terms)} \\
&= \cos(\pi) + i \sin(\pi) && \text{(Taylor series for $\cos(x)$ and $\sin(x)$)} \\
&= -1 && \text{(simplifying the previous expression).}
\end{aligned}$$

There are four columns in this example. The first column is right justified (and contains only $e^{i\pi}$). The second column is left justified, lining up the '=' signs. The third column (right justified) is empty because I want the explanation for each line to be lined up on the left.

1.4 Labels, links, and references

The label-reference mechanic is exceptionally useful tool for referencing the numbered items in your document. The number of any theorem, definition, equation, etc. is stored using `\label{[name]}` following the declaration of any numbered item, and can be recalled anywhere in the document using `\eqref{[name]}` for equations, and `\ref{[name]}` for all other items.

```
\begin{thm} \label{th:example theorem label}
Theorem \ref{th:example theorem label} is self-referential.
\end{thm}
\begin{align} \label{eq:example equation label}
\text{This is equation \eqref{eq:example equation label}.}
\end{align}
```

Theorem 1.4.1. *Theorem 1.4.1 is self-referential.*

$$\text{This is equation (1.2).} \tag{1.2}$$

The label-reference system is the preferred way to recall numbered items because the numbering will remain consistent even if you move these items around in your document.

1.5 Fonts

1.5.1 Text fonts

	Input	Output
Default	`ABCabc123`	ABCabc123
Bold	`\textbf{ABCabc123}`	**ABCabc123**
Italics	`\textit{ABCabc123}`	*ABCabc123*
Small capitals	`\textsc{ABCabc123}`	ABCabc123
Typewriter	`\texttt{ABCabc123}`	`ABCabc123`

1.5.2 Math fonts

	Input	Output
Default	`ABCabc123`	$ABCabc123$
Roman	`\mathrm{ABCabc123}`	$\mathrm{ABCabc123}$
Bold	`\mathbf{ABCabc123}`	$\mathbf{ABCabc123}$
Italics	`\mathit{ABCabc123}`	$\mathit{ABCabc123}$
Typewriter	`\mathtt{ABCabc123}`	$\mathtt{ABCabc123}$
Blackboard bold[1]	`\mathbb{ABC}`	$\mathbb{ABC}$
Blackboard bold (more)	`\mathbbm{abc12}`	$\mathbbm{abc12}$
Calligraphic	`\mathcal{ABC}`	$\mathcal{ABC}$
Euler script	`\EuScript{ABC}`	$\EuScript{ABC}$
Fraktur	`\mathfrak{ABCabc123}`	$\mathfrak{ABCabc123}$
Script	`\mathscr{ABC}`	$\mathscr{ABC}$

Note that some fonts are only available for certain characters.

[1] How else would you write in bold on a blackboard?

1.5.3 Text in math mode

Text can be written in math mode using `\text{[text]}` or `\atext{[text]}`[†]. These commands are particularly useful when it is inconvenient or impossible to leave math mode.

Input:

```
\begin{align*}
\sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}
\atext{is the Euler product for $\zeta(s)$ (if $\Re(s) > 1$).}
\end{align*}
```

Output:

$$\sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \quad \text{is the Euler product of } \zeta(s) \ (\Re(s) > 1).$$

1.5.4 Key words

Key words should be highlighted using `\emph`. Visually, this has the same effect as italicizing the word (or romanizing, if the word appears in italicized text).

Input: `It's time to \emph{sing}.`

Output: It's time to *sing*.

Input:

```
\textit{Bacon pancakes, makin' bacon pancakes. \\
Take some bacon and I put it in a pancake. \\
Bacon pancakes, that's what it's gonna make. \\
Ba- con pan- caaake \dotso in \emph{New York}}.
```

Output:

Bacon pancakes, makin' bacon pancakes.
Take some bacon and I put it in a pancake.
Bacon pancakes, that's what it's gonna make.
Ba- con pan- caaake . . . in New York.

https://www.youtube.com/watch?v=cUYSGojUuAU

1.6 Symbols

This is by no means an exhaustive list of symbols and commands. The intension is to provide a searchable reference of frequently used symbols. Nearly all of these commands are standard, but a few of the commands are my own (so you will only have access to these commands if you are using my `.tex` file). I have marked those commands with a dagger ([†]).

Unless otherwise stated, the commands are only available in math mode (the purple text must be placed between dollar signs to be displayed properly).

1.6.1 Algebraic expressions

Example		Translation
`a + b`	$a+b$	a plus b
`a - b`	$a-b$	a minus b
`a \cdot b`	$a \cdot b$	a times b (uncommon; in most cases we simply write ab)
`a \times b`	$a \times b$	a times b (very uncommon)
`\frac{a}{b}`	$\frac{a}{b}$	a over (divided by) b
`a/b`	a/b	a over (divided by) b (preferred for in-line expressions when it improves readability)
`a^{n}`	a^{n}	a to the n
`a^{n^{k^{13}}}`	$a^{n^{k^{13}}}$	(nested superscripts)
`a_{n}`	a_{n}	a sub n
`a_{n_{k_{13}}}`	$a_{n_{k_{13}}}$	(nested subscripts)
`\sqrt{a}`	$\sqrt{a}$	square root of a
`\sqrt[n]{a}`	$\sqrt[n]{a}$	n-th root of a
`|a|`	$\lvert a\rvert$	absolute value of a
`\pm a`	$\pm a$	plus or minus a
`\binom{a}{b}`	$\binom{a}{b}$	a choose b
`a \pmod{p}`	$a \pmod{p}$	a mod p (a modulo p)

1.6.2 Algebraic relations

Input	Output	Example		Translation
`=`	$=$	`a = b`	$a=b$	a is equal to b
`<`	$<$	`a < b`	$a<b$	a is less than b
`\le`	$\le$	`a \le b`	$a \le b$	a is less than or equal to b
`>`	$>$	`a > b`	$a>b$	a is greater than b
`\ge`	$\ge$	`a \ge b`	$a \ge b$	a is greater than or equal to b
`\equiv`	$\equiv$	`a \equiv b`	$a \equiv b$	a is congruent to b (generally used with `\pmod`)
`\sim`	$\sim$	`a \sim b`	$a \sim b$	a is related to b
`\mid`	$\mid$	`a \mid b`	$a \mid b$	a divides b

Many of these commands can be negated using the command `\not`. Note that some symbols have special commands for negation.

Input	Output
`\ne`	$\ne$
`\not <`	$\not<$
`\not\le`	$\not\le$
`\not >`	$\not>$
`\not\ge`	$\not\ge$
`\not\equiv`	$\not\equiv$
`\nsim`	$\nsim$
`\nmid`	$\nmid$

1.6.3 Braces, Brackets, Parentheses, etc.

Input	Output
`(a,b)`	(a,b)
`[a,b]`	$[a,b]$
`|a,b|`	$\|a,b\|$
`\\|a,b\\|`	$\\|a,b\\|$
`\{a,b\}`	$\{a,b\}$
`\langle a,b \rangle`	$\langle a,b\rangle$
`\Floor{\frac{a}{b}}`†	$\left\lfloor \frac{a}{b} \right\rfloor$

Resizing delimiters

These symbols can be automatically resized using the `\left` and `\right` commands.

Input: `\left\{ \Floor{ \left( a^n \right)^{n^2} }^{\frac{1}{2n-1}} \right\}`
Output:

$$\left\{ \left\lfloor \left(a^n \right)^{n^2} \right\rfloor^{\frac{1}{2n-1}} \right\}$$

The `\left` and `\right` commands do not need to take the same delimiters, but they do have to be paired. A period `.` can be used to produce one-sided delimiters.

Input:

```
\begin{align*}
\int_a^b x\,dx = \left. \frac{x^2}{2} \right|_a^b
\end{align*}
```

Output:

$$\int_a^b x\,dx = \left. \frac{x^2}{2} \right|_a^b$$

1.6.4 Dots

	Command	Example	
Comma separated lists	`\dotsc`	`1, 2, 3, \dotsc, 9`	$1,2,3,\dotsc,9$
Lower dots	`\ldots`	`1, 2, 3,\ldots,9`	$1,2,3,\ldots,9$
Binary expressions	`\dotsb`	`1 + 2 + \dotsb + 9`	$1+2+\dotsb+9$
Centered dots	`\cdots`	`1+2+\cdots+9`	$1+2+\cdots+9$
Multiplication (binary)	`\cdot`	`a \cdot b`	$a\cdot b$
Multiplication	`\dotsm`	`1 \cdot 2 \cdot 3 \dotsm 9`	$1\cdot 2\cdot 3\dotsm 9$
Otherwise [wide ellipsis]	`\dotso`	`a, b, c, \dotso, z`	a, b, c, . . . , z
Vertical dots	`\vdots`		$\vdots$
Diagonal dots	`\ddots`		$\ddots$

1.6.5 Functional notation

Input	Output	Example		Translation
`\to`	$\to$	`f \colon A \to B`	$f\colon A \to B$	f is a map from A to B
`\into`†	$\hookrightarrow$	`f \colon A \into B`	$f\colon A \hookrightarrow B$	f is an injective map from A to B (f maps A into B)
`\onto`†	$\twoheadrightarrow$	`f \colon A \onto B`	$f\colon A \twoheadrightarrow B$	f is a surjective map from A to B (f maps A onto B)
`\isom`†	$\xrightarrow{\sim}$	`f \colon A \isom B`	$f\colon A \xrightarrow{\sim} B$	f is a bijective map from A to B (f is an isomorphism from A to B)
`\circ`	$\circ$	`f \circ g`	$f \circ g$	f composed with g
`\mapsto`	$\mapsto$	`x \mapsto f(x)`	$x \mapsto f(x)$	x is mapped to $f(x)$

Defining functions:

Input: `f \colon A \to B`

Output: $f\colon A \to B$

Input:

```
\begin{align*}
f \colon A &\to B \\
x &\mapsto f(x)
\end{align*}
```

Output:

$$\begin{aligned} f\colon A &\to B \\ x &\mapsto f(x) \end{aligned}$$

Remark 1.6.1. Use `\colon` (not `:`) for proper spacing.

Input:

```
\begin{align*}
f \colon \mathbb{R} &\to \mathbb{R} \\
x &\mapsto x^2
\end{align*}
```

Output:

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

Translation: f is the function from the real numbers to the real numbers that maps x to x^2. (In other words, $f(x) = x^2$.)

Input:

```
\begin{align*}
f \colon \mathbb{Z} &\to \mathbb{F}_p \\
a &\mapsto a \bmod p
\end{align*}
```

Output:

$$\begin{aligned} f\colon \mathbb{Z} &\to \mathbb{F}_p \\ a &\mapsto a \bmod p \end{aligned}$$

Translation: f is the function from the integers to the finite field of order p that maps a to the reduction of a modulo p.

1.6.6 Greek alphabet

Math mode commands for Greek letters.

	Capital	Lowercase	Variant
A, α	`\mathrm{A}`	`\alpha`	
B, β	`\mathrm{B}`	`\beta`	
Γ, γ	`\Gamma`	`\gamma`	
Δ, δ	`\Delta`	`\delta`	
E, ϵ, ε	`\mathrm{E}`	`\epsilon`	`\varepsilon`
Z, ζ	`\mathrm{Z}`	`\zeta`	
H, η	`\mathrm{H}`	`\eta`	
Θ, θ, ϑ	`\Theta`	`\theta`	`\vartheta`
I, ι	`\mathrm{I}`	`\iota`	
K, κ	`\mathrm{K}`	`\kappa`	
Λ, λ	`\Lambda`	`\lambda`	
M, μ	`\mathrm{M}`	`\mu`	
N, ν	`\mathrm{N}`	`\nu`	
Ξ, ξ	`\Xi`	`\xi`	
O, o	`\mathrm{O}`	`\mathrm{o}`	
Π, π, ϖ	`\Pi`	`\pi`	`\varpi`
P, ρ, ϱ	`\mathrm{P}`	`\rho`	`\varrho`
$\Sigma, \sigma, \varsigma$	`\Sigma`	`\sigma`	`\varsigma`
T, τ	`\mathrm{T}`	`\tau`	
Υ, υ	`\Upsilon`	`\upsilon`	
Φ, ϕ, φ	`\Phi`	`\phi`	`\varphi`
X, χ	`\mathrm{X}`	`\chi`	
Ψ, ψ	`\Psi`	`\psi`	
Ω, ω	`\Omega`	`\omega`	

1.6.7 Large symbols

Symbols, as they appear in the `align` environment (or other similar environments). These symbols will be displayed differently if they are used in-line.

Input	Output	Example		Translation
`\sum`	$\sum$	`\sum_{n=1}^{\infty} n^{-s}`	$\displaystyle\sum_{n=1}^{\infty} n^{-s}$	the sum, from 1 to infinity, of n to the minus s
`\prod`	$\prod$	`\prod_{k=1}^{n} k`	$\displaystyle\prod_{k=1}^{n} k$	the product of the first n natural numbers
`\bigcup`	$\bigcup$	`\bigcup_{i=1}^{\infty} A_i`	$\displaystyle\bigcup_{i=1}^{\infty} A_i$	the union, from 1 to infinity, of the A_i
`\bigcap`	$\bigcap$	`\bigcap_{i=1}^{n} A_i`	$\displaystyle\bigcap_{i=1}^{n} A_i$	the intersection, from 1 to n, of the A_i

1.6.8 Logical symbols

Input	Output	Example		Translation
`\implies`†	$\implies$	`X \implies Y`	$X \implies Y$	X implies Y
`\impliedby`†	$\impliedby$	`X \impliedby Y`	$X \impliedby Y$	X is implied by Y
`\iff`†	$\iff$	`X \iff Y`	$X \iff Y$	X if and only if Y
`\contradiction`†	$\Rightarrow\Leftarrow$			contradiction
`\neg`	$\neg$	`\neg X`	$\neg X$	not X (negate X) also...
`\sim`	$\sim$	`\sim X`	$\sim X$	it is not true that X
`\land`	$\land$	`X \land Y`	$X \land Y$	X and Y
`\lor`	$\lor$	`X \lor Y`	$X \lor Y$	X or Y
`\forall`	$\forall$	`\forall a \in A`	$\forall a \in A$	for all elements in A
`\exists`	$\exists$	`\exists b \in B`	$\exists b \in B$	there exists a b in B
`!`	!	`\exists !b \in B`	$\exists !b \in B$	there exists a unique b in B

1.6.9 Quotation marks

The left quotation key is used to produce appropriate facing quotation marks.

Input	Output
`"Quotation marks always face right."`	”Quotation marks always face right.”
`‘Use the left-facing mark...’`	‘Use the left-facing mark...’
`‘‘...for left-facing quotation marks."`	“...for left-facing quotation marks.”

1.6.10 Set notation

Defining sets:

Input: `A = \{ [elements] : [conditions]\}`

Output: $A = \{[elements] : [conditions]\}$

Remark 1.6.2. Use `:` (not `\colon`) for proper spacing.

Example:

Input: `\mathbb{N} = \{ a \in \mathbb{Z} : a \ge 0 \}`

Output: $\mathbb{N} = \{a \in \mathbb{Z} : a \geq 0\}$

Translation: The natural numbers are the set of elements in the integers that are greater than or equal to zero. (The natural numbers are the set of non-negative integers.)

Example:

Input: `\mathbb{Q} = \left\{ \frac{a}{b} : a,b \in \mathbb{Z}, b \ne 0 \right\}`

Output: $\mathbb{Q} = \left\{ \frac{a}{b} : a,b \in \mathbb{Z}, b \ne 0 \right\}$

Translation: The rational numbers are the set of numbers of the form a/b, where a and b are integers, and b is non-zero.

Input	Output	Example		Translation
`\in`	$\in$	`a \in A`	$a \in A$	a is an element of A
`\subseteq`	$\subseteq$	`A \subseteq B`	$A \subseteq B$	A is a subset of B
`\subset`	$\subset$	`A \subset B`	$A \subset B$	A is a proper subset of B
`\supseteq`	$\supseteq$	`A \supseteq B`	$A \supseteq B$	A contains B (as a subset)
`\supset`	$\supset$	`A \supset B`	$A \supset B$	A contains B as a proper subset
`=`	$=$	`A = B`	$A = B$	A is equal to B
`\cong`	$\cong$	`A \cong B`	$A \cong B$	A is isomorphic to B
`\cup`	$\cup$	`A \cup B`	$A \cup B$	A union B
`\cap`	$\cap$	`A \cap B`	$A \cap B$	A intersect B
`-`	$-$	`A-B`	$A - B$	A (set) minus B (i.e. the elements of A that are not in B)
`|`	$\vert$	`|A|`	$\vert A\vert$	cardinality of A
`\mathscr P(A)`	$\mathscr{P}(A)$			the power set of A
`A^c`	A^c			the complement of A (also: `\overline{A}` $\overline{A}$)
`\emptyset`	$\emptyset$	`\{\} = \emptyset`	$\{\} = \emptyset$	$\{\}$ is the empty set (also: `\varnothing` $\varnothing$)

Negations:

Input	Output
`\notin`	$\notin$
`\not\subseteq`	$\not\subseteq$
`\not\subset`	$\not\subset$
`\not\supseteq`	$\not\supseteq$
`\not\supset`	$\not\supset$

1.6.11 Special symbols

Most of these commands are valid in both math mode and text mode.

Input	Output	Comment
`\#`	#	
`\$`	$	
`\%`	%	
`\^{}`	ˆ	text mode only
`\&`	&	
`\_`	_	
`\{`	{	
`\}`	}	
`\textgreater`	>	
`\textless`	<	
`\textbackslash`	\	
`\backslash`	\	math mode only

1.7 Color

1.7.1 Colored text and text boxes

Colored text is created using `{\color{[color]} [text] }` (also `\textcolor{[color]}{[text]}`). Here are a few key words for colors in the colors they generate.

blue	gray	Navy	red
Brown brown	Green green	Orange orange	Silver
Cyan cyan	Magenta magenta	pink	Yellow yellow
Gold	Maroon	Purple purple	YellowGreen

Capitalization matters![2]

Backgrounds are colored using `\colorbox{[color]}{[text]}`.

blue		gray		Navy		red	
Brown	brown	Green	green	Orange	orange	Silver	
Cyan	cyan	Magenta	magenta	pink		Yellow	yellow
Gold		Maroon		Purple	purple	YellowGreen	

And, of course, black and white.

The commands `\color`, `\textcolor`, and `\colorbox` work in both math and text mode.

1.7.2 Mixing colors

Shades of colors can be specified using the command `[color!#]`, where the number determines the percentage of that color used in the mix. When only one color is specified, the remainder of the mixture is white.

[2] Why do some names have two colors? It's a long story.
http://en.wikipedia.org/wiki/Cyan#Cyan_on_the_web_and_in_printing

Input	Output	% blue (box)	% white (box)
`\colorbox{blue!100}{\color{red!0}red}`	red	100	0
`\colorbox{blue!90}{\color{red!10}red}`	red	90	10
`\colorbox{blue!80}{\color{red!20}red}`	red	80	20
`\colorbox{blue!70}{\color{red!30}red}`	red	70	30
`\colorbox{blue!60}{\color{red!40}red}`	red	60	40
`\colorbox{blue!50}{\color{red!50}red}`	red	50	50
`\colorbox{blue!40}{\color{red!60}red}`	red	40	60
`\colorbox{blue!30}{\color{red!70}red}`	red	30	70
`\colorbox{blue!20}{\color{red!80}red}`	red	20	80
`\colorbox{blue!10}{\color{red!90}red}`	red	10	90
`\colorbox{blue!0}{\color{red!100}red}`	red	0	100

More than one color can be specified as inputs as long as the colors are separated by "percent". The last color is always used to fill out the mixture.

Input	Output	% Cyan	% Magenta
`\colorbox{Cyan!0!Magenta}{text}`	text	0	100
`\colorbox{Cyan!10!Magenta}{text}`	text	10	90
`\colorbox{Cyan!20!Magenta}{text}`	text	20	80
`\colorbox{Cyan!30!Magenta}{text}`	text	30	70
`\colorbox{Cyan!40!Magenta}{text}`	text	40	60
`\colorbox{Cyan!50!Magenta}{text}`	text	50	50
`\colorbox{Cyan!60!Magenta}{text}`	text	60	40
`\colorbox{Cyan!70!Magenta}{text}`	text	70	30
`\colorbox{Cyan!80!Magenta}{text}`	text	80	20
`\colorbox{Cyan!90!Magenta}{text}`	text	90	10
`\colorbox{Cyan!100!Magenta}{text}`	text	100	0

The reason why I put "percent" in quotation marks is because when mixing more than two colors, the numbers may not be what you expect.

Input	Output	% red	% blue	% yellow
`\colorbox{red!50!blue!66!yellow}{text}`	text	33	33	34
`\colorbox{red!33!blue!33!yellow}{text}`	text	11	22	67

Input	Output	% red	% blue	% white
`\colorbox{red!50!blue}{text}`	text	50	50	0
`\colorbox{red!50!blue!50}{text}`	text	25	25	50
`\colorbox{red!33!blue!33!white!33}{text}`	text	4	7	89

However if you want to be this specific about blending colors, you might as well use `rgb`, `html`, or one of the other standard conventions for specifying colors. (See the LaTeX Wikibook for details.)

[†]These commands are unique to my `.tex` file.