MATH 2001
CONDITIONALS, QUANTIFIERS, AND NEGATION

As stated previously, if-then statements are very common. Due to the nuance and variation of the English language, there are many ways to restate an if-then statement. Depending on the context, some phrasings may be more appropriate than others. We have already seen one example: the statement "if P, then Q" is equivalent to the statement "$P \Rightarrow Q$". Here are a few more equivalent forms.

Exercise 1. Consider the following "if P, then Q" statement.

If $\underbrace{x = 1}_{P}$, then $\underbrace{x^2 = 1}_{Q}$.

Complete each of the following statements so that they are equivalent to the if-then statement above.

a. ______________________________ if ______________________________.

b. ______________________________ only if ______________________________.

c. ______________________________ implies ______________________________.

d. ______________________________ is implied by ______________________________.

e. ______________________________ whenever ______________________________.

f. ______________________________ is necessary when ______________________________.

g. ______________________________ is sufficient for ______________________________.

As long as we can identify the P and Q parts in statements like the ones above, we can follow the rules to form the negations, contrapositives, and converses of the statements as necessary.

Exercise 2. For the statement "$P \Rightarrow Q$", the contrapositive, negation, and converse are

- Contrapositive:
- Negation:
- Converse:

For each/all/every: "For all P, we have Q" or "Q for all P". Another common phrase is the "for all" statement. For example: "$x^2 < 0$ for all $x \in \mathbb{R}$." (Equivalently, "for all $x \in \mathbb{R}$, we have $x^2 < 0$.")

There exists: "There exists P such that Q". The "for all" statement implies that the statement is always true. On the other end of the spectrum, the "there exists" statement implies that the statement is true in at least one case (but may be false otherwise). For example: "there exists $c \in \mathbb{Z}$ such that $3c = 7$."

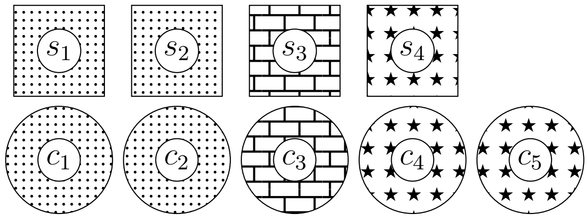
Of course, both my examples are false statements. So by negating these statements, we will have true statements. How do we negate "for all" and "there exists"?

- $\neg$("for all P, we have Q") = "there exists P such that $\neg Q$"
- $\neg$("there exists P such that Q") = "for all P, we have $\neg Q$"

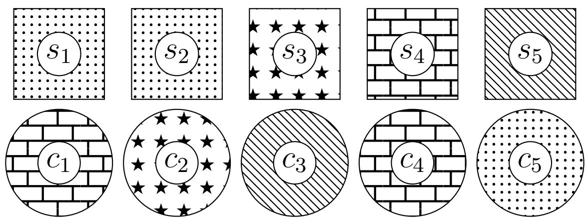
E.g. $\neg$("for all $x \in \mathbb{R}$, we have $x^2 < 0$") = "there exists $x \in \mathbb{R}$ such that $x^2 \geq 0$."

$\neg$("there exists $c \in \mathbb{Z}$ such that $3c = 7$") = "for all $c \in \mathbb{Z}$, we have $3c \neq 7$."

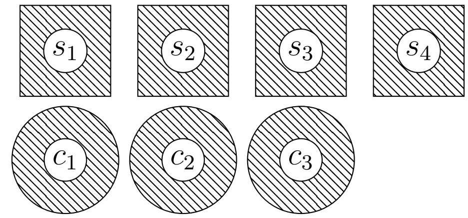
I.



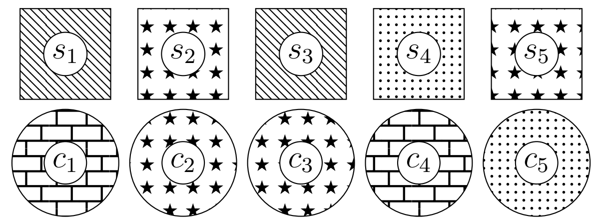
II.



III.



IV.



Let $P(s_i)$ and $P(c_i)$ denote the pattern on the i-th square or circle, respectively.

- For each of the following statements, write its contrapositive, negation, and converse where requested.
- In the neighboring boxes, identify the diagrams (**I, II, III, IV**) that satisfy each statement.

Satisfied by

1. If $i = j$, then $P(s_i) = P(c_j)$.

   Contrapositive:

   Negation:

   Converse:

2. The pattern $P(c_i)$ is “lines” whenever $P(s_i)$ is “lines”.

   Contrapositive:

   Negation:

   Converse:

3. For each i, there exists a j for which $P(c_i) = P(s_j)$.

   Negation:

4. $\square$ For each i, there exists a j for which $P(s_i) = P(c_j)$.

$\square$ Negation:

5. $\square$ There exists an i such that $P(s_j) = P(c_i)$ for all j.

$\square$ Negation:

6. $\square$ For all i and all j, we have $P(c_i) = P(s_j)$.

$\square$ Negation:

7. $\square$ There exists an i and there exists a j such that $P(c_i) = P(s_j)$.

$\square$ Negation:

8. $\square$ If there exists an i for which $P(c_i) = P(s_i)$, then there exists a $j \neq i$ such that $P(c_i) = P(c_j)$.

$\square$ Contrapositive:

$\square$ Negation:

$\square$ Converse:

9. $\square$ There exists an i for which $P(c_i)$ is "dots" only if there exists a j for which $P(s_j)$ is "dots".

$\square$ Contrapositive:

$\square$ Negation:

$\square$ Converse:

Upcoming deadlines:

- Due Friday, Mar 4: final draft of proof 4, first draft of proof 7.
- Due Monday Mar 7: final draft of proof 5, final draft of proof 6.
- Due Wednesday Mar 9: final draft of proof 7, first draft of proof 8.

Exercise 3. Complete each of the following statements using the following clauses.

P: It rains heavily. Q: The streams flood their banks.

(You may need to alter how the clauses are phrased.)

a. If ______________________________, then ______________________________.

b. ______________________________ if ______________________________.

c. ______________________________ only if ______________________________.

d. ______________________________ implies ______________________________.

e. ______________________________ implied by ______________________________.

f. ______________________________ whenever ______________________________.

g. ______________________________ necessary ______________________________.

h. ______________________________ sufficient ______________________________.