MATH 2001
TOPOLOGY

Upcoming deadlines:

- Due Monday, Mar 14: final draft of proof 8, first draft of proof 9.
- Due Wednesday, Mar 16: first draft of proof 10.
- Due Friday, Mar 18: final draft of proof 9.

Definition. Let A be a set, and let $\mathcal{T}$ be a set of subsets of A. The set $\mathcal{T}$ is a *topology on A* if $\mathcal{T}$ satisfies the following properties.

(1) The sets $\varnothing$ and A are elements of $\mathcal{T}$.

(2) The set $\mathcal{T}$ is closed under arbitrary union: if $S \subseteq \mathcal{T}$, then

$$\bigcup_{U \in S} U \in \mathcal{T}.$$

That is, the union of any number of elements in $\mathcal{T}$ (finite or infinite) is an element of $\mathcal{T}$.

(3) The set $\mathcal{T}$ is closed under finite intersection: if $S \subseteq \mathcal{T}$ and $|S| < \infty$, then

$$\bigcap_{U \in S} U \in \mathcal{T}.$$

That is, the intersection of finitely many elements in $\mathcal{T}$ is an element of $\mathcal{T}$.

Exercise 1. Let $A = \{a, b, c, d\}$. For each set $\mathcal{T}$, determine if $\mathcal{T}$ is a topology on A. If $\mathcal{T}$ is not a topology, find the smallest set containing $\mathcal{T}$ that is a topology.

a. $\mathcal{T} = \{\varnothing, A\}$

b. $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}, A\}$

c. $\mathcal{T} = \{\varnothing, \{a\}, \{b\}, A\}$

d. $\mathcal{T} = \{\varnothing, \{a, b\}, \{c, d\}, A\}$

e. $\mathcal{T} = \{\varnothing, \{a\}, \{a, b, c\}, \{c, d\}, A\}$

Exercise 2. Let A be a set. Prove that $\mathscr{P}(A)$ is a topology on A.

Remark. The definition of topology is often a difficult definition to satisfy because it is a lot of work to check that every union and every finite intersection is contained in the topology.

Definition. Let A be a set, and let $\mathcal{B}$ be a set of subsets of A. The set $\mathcal{B}$ is a *basis for a topology on* A if it satisfies the following properties.

(1) For each $x \in A$, there exists $B \in \mathcal{B}$ such that $x \in B$.

(2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$, and $B_3 \subseteq B_1 \cap B_2$.

Exercise 3. Let $A = \{a, b, c, d\}$. For each set $\mathcal{B}$, determine if $\mathcal{B}$ is a basis for a topology on A. If $\mathcal{B}$ is not a topology, explain why it violates the definition.

a. $\mathcal{B} = \{A\}$

b. $\mathcal{B} = \{\{a\}, \{a, b\}, \{a, b, c\}, A\}$

c. $\mathcal{B} = \{\{a\}, \{b\}, \{c, d\}\}$

d. $\mathcal{B} = \{\{b, c\}, \{a, b, c\}, \{b, c, d\}\}$

e. $\mathcal{B} = \{\{a\}, \{a, b, c\}, \{c, d\}\}$

Exercise 4. Prove that $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$.

Exercise 5. Prove that $\mathcal{B} = \{U \subseteq \mathbb{R} : |\overline{U}| < \infty\}$ is a basis for a topology on $\mathbb{R}$.

Definition. Let A be a set, and let $\mathcal{B}$ be a basis for a topology on A. The *topology* $\mathcal{T}$ *generated by* $\mathcal{B}$ is the set of all $U \subseteq A$ that satisfy: if $x \in U$, then there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. That is,

$$\mathcal{T} = \{U \subseteq A : \text{for each } x \in U, \text{ there exists } B \in \mathcal{B} \text{ such that } x \in B \text{ and } B \subseteq U\}.$$

Exercise 6. Prove that if A is a set, $\mathcal{B}$ is a basis for a topology on A, and $\mathcal{T}$ is the topology generated by $\mathcal{B}$, then $\mathcal{T}$ is a topology on A.

Exercise 7. Prove that if A is a set and $\mathfrak{T}$ is a topology on A, then there exists a basis for a topology $\mathcal{B}$ that generates $\mathfrak{T}$.