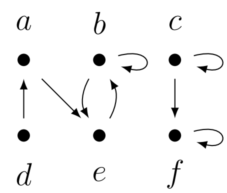
MATH 2001
RELATIONS

Definition 1. Let A be a set. The set R is a *relation* on A if $R \subseteq A \times A$.

Definition 2. Let A be a set and let R be a relation on A. Then

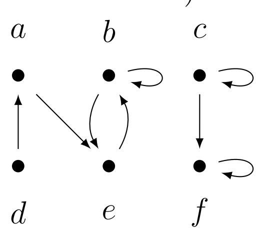
- R is *reflexive* if $(a, a) \in R$ for each $a \in A$;
- R is *symmetric* if $(a, b) \in R$ implies that $(b, a) \in R$;
- R is *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.

Example 3. Let $A = \{a, b, c, d, e, f\}$ and consider the relation R given by the following diagram.

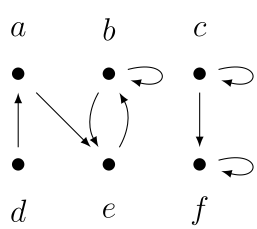


a.) What is R? (Write out the elements of R in set notation.)

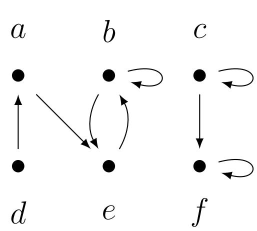
b.) What additional elements would have to be included for R to be reflexive? (Feel free to draw them in as well.)



c.) What additional elements would have to be included for R to be symmetric?



d.) What additional elements would have to be included for R to be transitive?



Example 4. Each of the following is a relation on $\mathbb{Z}$: $=, \neq, \leq, |, \nmid$. For each relation, determine whether it is reflexive, symmetric, and/or transitive. If a relation does not have a particular property, give an example illustrating why not.

Definition 5. A relation R is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Example 6. Which of the relations in Example 4 are equivalence relations?

Definition 7. Let A be a set, and let R be an equivalence relation on A. For any $a \in A$, the *equivalence class containing* a (denoted by $[a]$) is the set of all elements in A that are related to a. That is,

$$[a] = \{b \in A : (a, b) \in R\}.$$

Example 8. Let $A = \mathbb{Z}$, and let R be the relation on A defined by

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b = 2c \text{ for some } c \in \mathbb{Z}\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.

Example 9. Let $A = \mathbb{Z}^2 - \{(0, 0)\}$, and let R be the relation on A defined by

$$R = \{((a, b), (c, d)) : ad - bc = 0\}.$$

Prove that R is an equivalence relation on $\mathbb{Z}$, then describe the equivalence classes.