# MATH 2001
# PROOFS BY INDUCTION

Some of the problems from yesterday's sheet are included on this one.

**Arithmetic.**

1.) Prove that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for every integer $n \geq 1$.

2.) Prove that $\displaystyle\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for every integer $n \geq 0$.

3.) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers where $a_1 = 3$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for each integer $n \geq 4$. Prove that $1 \leq \dfrac{a_n}{a_{n-1}} \leq 2$ for each $n \geq 4$.

**Sets.**

4.) Suppose that $A_1, A_2, A_3, A_4, \ldots$ is an infinite sequence of non-empty, nested sets. That is,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

and each $A_i$ is non-empty.

(a) Prove that $\bigcap_{i=1}^{N} A_i$ is non-empty for every $N \in \mathbb{N}$.

(b) Is the set $\bigcap_{i=1}^{\infty} A_i$ empty or non-empty?

If possible, give an example where $\bigcap_{i=1}^{\infty} A_i$ is empty.

If possible, give an example where $\bigcap_{i=1}^{\infty} A_i$ is non-empty.

5.) (a) Prove the finite versions of de Morgan's laws by induction.

(i) $\displaystyle\overline{\bigcap_{1\leq i\leq N} A_i} = \bigcup_{1\leq i\leq N} \overline{A_i}.$ (ii) $\displaystyle\overline{\bigcup_{1\leq i\leq N} A_i} = \bigcap_{1\leq i\leq N} \overline{A_i}.$

(b) Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of sets. Explain why induction cannot be used to prove the next two statements.

(i) $\displaystyle\overline{\bigcap_{i\in\mathbb{N}} A_i} = \bigcup_{i\in\mathbb{N}} \overline{A_i}.$ (ii) $\displaystyle\overline{\bigcup_{i\in\mathbb{N}} A_i} = \bigcap_{i\in\mathbb{N}} \overline{A_i}.$

Prove these "infinite" versions of de Morgan's laws using a method other than induction. In fact, these statements are true for arbitrary unions and intersections. Your proofs for (i) and (ii) should also work for (iii) and (iv). (You don't need to write the proofs twice.)

(iii) $\displaystyle\overline{\bigcap_{\lambda\in\Lambda} A_i} = \bigcup_{\lambda\in\Lambda} \overline{A_i}$ (iv) $\displaystyle\overline{\bigcup_{\lambda\in\Lambda} A_i} = \bigcap_{\lambda\in\Lambda} \overline{A_i}.$

**Topology.**

6.) (a) Prove that a finite intersection of open sets is open. That is, prove that $\bigcap_{i=1}^{N} A_i$ is open if each $A_i$ is open (and $N \in \mathbb{N}$). (c.f. Proof 13)

(b) Give an example of an infinite intersection of open sets that is not open. (Hint: Proof ??)

(c) Prove that a finite union of closed sets is closed. That is, prove that $\bigcup_{i=1}^{N} A_i$ is closed if each $A_i$ is closed. (Hint: de Morgan's laws.)

(d) Give an example of a infinite intersection of closed sets that is not closed.

7.) (a) Prove that a union of arbitrarily many open sets is open. That is, prove that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is open if each $A_\lambda$ is open.

(b) Prove that the intersection of arbitrarily many closed sets is closed.

**Functions.**

8.) Let $f \colon A \to A$ be a bijection. Prove that $f^n \colon A \to A$ is a bijection. [Hint: functions worksheet.]

Here, $f^n$ denotes the $n$-fold composition of $f$ with itself. That is, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc. In general, $f^{n+1}(x) = f(f^n(x))$.

9.) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function with the property that $|f(b) - f(a)| < \frac{1}{2}|b - a|$ for all $a, b \in \mathbb{R}$. Prove that $|f^n(b) - f^n(a)| < \frac{1}{2^n}|b - a|$ for all $a, b \in \mathbb{R}$.

**Critique the following proof.**

**Theorem.** *All horses are the same color.*

*Proof.* We prove the result by induction by showing that in every collection of $n$ horses, all the horses have the same color.

For the base case, let $n = 1$. If there is only one horse, then there is only one color (e.g. blue). Thus in every set of 1 horses, every horse in that set has the same color as all other horses in that set.

For the induction step, suppose there is an $n$ for which in every collection of $n$ horses, all the horses have the same color. Select any $n$ of the horses on Earth, then as we now have $n$ horses, all of these horses have the same color. For simplicity, let's say that they are all blue. Remove one horse from this collection (leaving $n-1$ blue horses) and add to this set any horse on Earth that was not previously selected. We now have a new set of $n$ horses, and therefore they all have the same color. In particular, the newly selected horse is blue as it has the same color as the other $n - 1$ horses, which we already know to be blue. Bring back the blue horse that we initially removed, and we now have $n + 1$ blue horses. This completes the induction step.

Finally, as the number of horses on Earth is finite, we can induct up to that number and conclude that all horses are the same color. □