2.3 — Matrix products
University of Massachusetts Amherst
Math 235 — Spring 2014

Matrix multiplication: Recall that if A is an $\ell \times m$ matrix, and B is an $m \times n$ matrix, and we write

$$\begin{bmatrix} — & \vec{r}_1 & — \\ — & \vec{r}_2 & — \\ & \vdots & \\ — & \vec{r}_\ell & — \end{bmatrix}, \quad B = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{bmatrix},$$

where $\vec{r}_1, \ldots, \vec{r}_\ell$ are row vectors and $\vec{c}_1, \ldots, \vec{c}_n$ are column vectors, then the product

$$AB = \begin{bmatrix} \vec{r}_1\vec{c}_1 & \vec{r}_1\vec{c}_2 & \cdots & \vec{r}_1\vec{c}_n \\ \vec{r}_2\vec{c}_1 & \vec{r}_2\vec{c}_2 & \cdots & \vec{r}_2\vec{c}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_\ell\vec{c}_1 & \vec{r}_\ell\vec{c}_2 & \cdots & \vec{r}_\ell\vec{c}_n \end{bmatrix}$$

is an $\ell \times n$ matrix.

Note: If A is an $\ell \times a$ matrix and B is a $b \times n$ matrix, then the product AB is defined only if

Example 1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Which products are defined?

(i) AC (ii) CB (iii) AB (iv) BA

Compute the product, if defined.

Example 2. Find all matrices that commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Example 3. Find a 2×2 matrix A such that $A^2 = I$ and all entries of A are non-zero integers.

Properties of matrix multiplication:

(1) Identity: $IA = AI$
(2) Associative: $A(BC) = (AB)C$
(3) Distributive: $A(B + C) = AB + AC$
(4) Scalar multiplication: $A(kB) = (kA)B = k(AB)$.

Additional exercises

(1) Find all matrices that commute with $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
(2) Find a 2×2 matrix A such that $A^2 = A$ and all entries of A are non-zero.
(3) Find all matrices A such that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.