3.1 — Image and kernel of a linear transformation
University of Massachusetts Amherst
Math 235 — Spring 2014

Recall that if $\vec{v}_1, \dots, \vec{v}_m$ are vectors and $a_1, \dots, a_m$ are scalars, then the linear combination of these vectors with these scalars as coefficients is

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m.$$

Definition 1. Let $\vec{v}_1, \dots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of these vectors is called their *span*, for which we write $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_m)$.

Example 2.

(i) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. What is $\operatorname{span}(\vec{v}_1)$? Express the vector $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v}_1$.

(ii) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. What is $\operatorname{span}(\vec{v}_1, \vec{v}_2)$? Express $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2$.

(iii) What is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. Write $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ as a linear combination of these vectors.

Definition 3. The *image* of a function f is the set of all outputs of f. That is,

$$\text{im}(f) = \{f(x) \colon x \in \text{domain}(f)\}.$$

Theorem 4. *Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ with associated matrix A. The image of T is the span of the columns of A.*

Proof. We want to show that $\text{im}(T) = \text{span}(\text{ columns of } A \text{ })$. In other words, we want to prove that $\vec{b} \in \text{im}(T)$ if and only if $\vec{b} \in \text{span}(\text{ columns of } A \text{ })$.

$\square$

Example 5. Describe the image of the linear transformation represented by the matrix A, when

$$(a) \quad A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (b) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (c) \quad A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 6. Is $\vec{b}$ in the image of the linear transformation given by the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$?

If so, express $\vec{b}$ as a linear combination of the columns of A.

Theorem 7. *The image of a linear transformation T (from $\mathbb{R}^m$ to $\mathbb{R}^n$) has the following properties:*
(a) The zero vector $\vec{0}$ in $\mathbb{R}^n$ is in the image of T.
(b) The image of T is closed under addition: If $\vec{b}_1$ and $\vec{b}_2$ are in the image of T, then so is $\vec{v}_1 + \vec{v}_2$.
(c) The image of T is closed under scalar multiplication: If $\vec{v}$ is in the image of T, then so is $k\vec{v}$ for any scalar k.

Proof.

□

Example 8. Let A be an $n \times n$ matrix. Show that $\operatorname{im}(A^2)$ is a subset of $\operatorname{im}(A)$. That is, show that if $\vec{b} \in \operatorname{im}(A^2)$, then $\vec{b} \in \operatorname{im}(A)$.

Definition 9. The *kernel* (or null space) of a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is the set of all solutions to the equation $T(\vec{x}) = \vec{0}$. That is,

$$\ker(T) = \operatorname{Nul}(T) = \{x \in \mathbb{R}^m \colon T(\vec{x}) = \vec{0}\}$$

Example 10. Find the kernel of the linear transformation $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$.

Example 11. Describe the image and kernel of $\text{proj}_{\vec{v}}(\vec{x})$ when $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Theorem 12. *Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.*
(a) The zero vector $\vec{0}$ in $\mathbb{R}^m$ is in the kernel of T.
(b) The kernel of T is closed under addition.
(c) The kernel of T is closed under scalar multiplication.

Theorem 13.
(a) If A is an $n \times m$ matrix, then $\ker(A) = \{\vec{0}\}$ if and only if $\text{rank}(A) = m$.
(b) If A is an $n \times m$ matrix and $\ker(A) = \{\vec{0}\}$, then $m \leq n$. If $m > n$, then then $\ker(A)$ contains non-zero vectors.
(c) If A is a square matrix, then $\ker(A) = \{\vec{0}\}$ if and only if A is invertible.

Proof.

□

Theorem 14. *Let A be an $n \times n$ matrix. The following statements are equivalent. (If one statement is true, then all the statements are true; if one statement is false, then all are false.)*
(a) A is invertible.
(b) The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b}$ in $\mathbb{R}^n$.
(c) $\text{rref}(A) = I_n$.
(d) $\text{rank}(A) = n$.
(e) $\text{im}(A) = \mathbb{R}^n$.
(f) $\ker(A) = \{\vec{0}\}$.

ADDITIONAL EXERCISES

(1) Prove Theorem 11.
(2) Give spanning sets for the image and the kernel of the matrix A when

$$(a) \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad (b) \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (c) \quad A = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) Suppose that A is a square matrix and $\ker(A^2) = \ker(A^3)$. Is it true that $\ker(A^3) = \ker(A^4)$?