1. Vector spaces

(1) Define *vector space*: V is a vector space if ...

(2) Define *vector subspace*: W is a subspace of a vector space V if ...

(3) Determine which of the following sets are vector subspaces.

(a) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

(b) $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$

(c) $\left\{ \begin{bmatrix} 1-a \\ a-6b \\ 2b+a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(d) $\left\{ \begin{bmatrix} 2b+3c \\ b \\ c \end{bmatrix} : b, c \in \mathbb{R} \right\}$

(e) $\{p(t) \in P_2 \colon p(2) = 0\}$, ($P_2$ is the set of polynomials of degree at most 2).

(f) $\{p(t) \in P_2 \colon \int_0^1 p(t)\,dt = 0\}$

(g) $\{p(t) \in P_2 \colon p(t) = p(-t)\}$

(h) $\{p(t) \in P_2 \colon -p(t) = p(-t)\}$

(i) $\left\{ A \in \mathbb{R}^{3\times 3} \colon \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \ker(A) \right\}$

(j) $\{B \in \mathbb{R}^{3\times 3} \colon B \text{ is in reduced row-echelon form}\}$

(k) $\left\{ A \in \mathbb{R}^{2\times 2} \colon \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = A \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

(l) The set of infinite sequences of the form $(a, a+k, a+2k, a+3k, \dots)$ where a and k are constants.

(m) The set of infinite sequences of the form $(a, ak, ak^2, ak^3, \dots)$ where a and k are constants.

(4) Find a basis and determine the dimension of each of the following subspaces.

(a) The space of all polynomials $f(t) \in P_3$ such that $f(1) = 0$ and $\int_{-1}^1 f(t)\,dt = 0$.

(b) The space of all lower triangular 2×2 matrices.

(c) The space of all 2×2 matrices A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(5) Find all solutions of the differential equation $f''(x) + 8f'(x) - 20f(x) = 0$. (See Example 3 in the 4.2 notes.)

Selected answers:

(3) Recall that a space W is a vector subspace if

- W contains the zero element.
- W is closed under addition: if f and g are in W, then $f + g$ is in W.
- W is closed under scalar multiplication: if f is in W, then kf is in W for every scalar k.

(a) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ is not a subspace because it is not closed under scalar multiplication: the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in W, but $-\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is not in W.

Note that this is the only reason why W fails to be a vector space. W contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the zero element, and W is closed under addition: if $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are in W, then $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is in W since $x_1 + x_2 \geq 0$ and $y_1 + y_2 \geq 0$.

(b) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$ is not a vector subspace because it is not closed under addition: the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are in W, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not in W.

This space does contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the zero element, and this space is closed under scalar multiplication: if $\begin{bmatrix} x \\ y \end{bmatrix}$ is in W, then $k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$ is in W since $(kx)(ky) = k^2xy \geq 0$. ($xy \geq 0$ since $\begin{bmatrix} x \\ y \end{bmatrix}$ is in W).

(c) $W = \left\{ \begin{bmatrix} 1-a \\ a-6b \\ 2b+a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is not a vector subspace because, for example, it does not contain the zero element: there are no real numbers a and b such that $\begin{bmatrix} 1-a \\ a-6b \\ 2b+a \end{bmatrix} = \begin{bmatrix} 0, 0, 0 \end{bmatrix}$.

(f) $W = \{p(t) \in P_2 \colon \int_0^1 p(t)\,dt = 0\}$ is a vector subspace: 1) W contains the zero function $p_0(t) = 0 + 0t + 0t^2$ since $\int_0^1 0 + 0t + 0t^2\,dt = 0$. 2) W is closed under addition: if $p_1(t)$ and $p_2(t)$ are in W, then $p_1(t) + p_2(t)$ is in W since $\int_0^1 p_1(t) + p_2(t)\,dt = \int_0^1 p_1(t)\,dt + \int_0^1 p_2(t)\,dt = 0 + 0 = 0$. 3) W is closed under scalar multiplication: if $p(t)$ is in W, then $kp(t)$ is in W since $\int_0^1 kp(t)\,dt = k\int_0^1 p(t)\,dt = k \cdot 0 = 0$.

(i) $W = \left\{ A \in \mathbb{R}^{3\times3} \colon \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \ker(A) \right\}$ is a vector subspace: 1) W contains the zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the kernel of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. 2) W is closed under addition: if A_1 and A_2 are in W, then $A_1 + A_2$ is in W since $(A_1 + A_2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + A_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. 3) W is closed under scalar multiplication: if A is in W, then kA is in W since $(kA)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = kA\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = k\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(l) $W = \{(a, a+k, a+2k, a+3k, \ldots)\}$ is a vector space: 1) W contains the sequence $(0, 0, 0, \ldots)$ $(a = k = 0)$. 2) W is closed under addition: if $(a_1, a_1 + k_1, a_1 + 2k_1, \ldots)$ and $(a_2, a_2 + k_2, a_2 + 2k_2, \ldots)$ are in W, then the sum of the sequences is in W since $(a_1, a_1 + k_1, a_1 + 2k_1, \ldots) + (a_2, a_2 + k_2, a_2 + 2k_2, \ldots) = (a_1 + a_2, a_1 + a_2 + k_1 + k_2, a_1 + a_2 + 2k_1 + 2k_2, \ldots) = (a, a+k, a+2k, \ldots)$ where $a = a_1 + a_2$ and $k = k_1 + k_2$.

(4) To find a basis, we need to write down a general element of the vector space and determine the number of free variables.

(a) A general polynomial of degree 3 has the form $f(t) = a + bt + ct^2 + dt^3$. Additionally, if $f(1) = 0$, then $a + b + c + d = 0$, and if $\int_{-1}^{1} f(t)\,dt = 0$, then

$$\begin{aligned} 0 &= \int_{-1}^{1} a + bt + ct^2 + dt^3\,dt = at + \frac{bt^2}{2} + \frac{ct^3}{3} + \frac{dt^4}{4}\Big|_{-1}^{1} \\ &= \left(a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}\right) - \left(-a + \frac{b}{2} - \frac{c}{3} + \frac{d}{4}\right) = 2a + \frac{2c}{3}. \end{aligned}$$

We must now solve the system of equations

$$\begin{aligned} 2a + \frac{2c}{3} &= 0 \\ a + b + c + d &= 0. \end{aligned}$$

Putting these in an augmented matrix, we have

$$\begin{bmatrix} 2 & 0 & 2/3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 2/3 & 1 & 0 \end{bmatrix},$$

hence $a = -\frac{c}{3}$ and $b = -\frac{2c}{3} - d$. Plugging these back into the equation for $f(t)$, we see that

$$f(t) = -\frac{c}{3} - \left(\frac{2c}{3} + d\right)t + ct^2 + dt^3 = c\left(-\frac{1}{3} - \frac{2}{3}t + t^2\right) + d\left(-t + t^3\right).$$

Thus f is a linear combination of the polynomials $t^3 - t$ and $t^2 - \frac{2}{3}t - \frac{1}{3}$. Moreover, these polynomials form a linearly independent set, and thus give a basis for this space: $\mathfrak{B} = \left\{t^3 - t, t^2 - \frac{2}{3}t - \frac{1}{3}\right\}$. There are two elements in the basis, so the dimension of this vector space is 2.

(c) A general 2×2 matrix is $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For A to satisfy the condition to be in the set, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$$

Hence $a = -b$ and $c = -d$, and

$$A = \begin{bmatrix} -b & b \\ -d & d \end{bmatrix} = b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

Thus a basis for this space is $\mathfrak{B} = \left\{\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\right\}$, and the dimension of this space is 2.

2. Linear transformations

(1) Define *linear transformation*: a function T from a vector space V to a vector space W is a linear transformation if ...

(2) Which of the transformations are linear? For those that are linear, determine whether they are isomophisms.

(a) $T(M) = M + I_2$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(b) $T(M) = 7M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(c) $T(M) = M^2$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(d) $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(e) $T(M) = M\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(f) $T(a+bi) = a$ from $\mathbb{C}$ to $\mathbb{C}$. ($\mathbb{C}$ is the set of complex numbers.)

(g) $T(a+bi) = b+ai$ from $\mathbb{C}$ to $\mathbb{C}$.

(h) $T(f(t)) = f(7)$ from P_2 to $\mathbb{R}$.

(i) $T(f(t)) = f''(t)f(t)$ from P_2 to P_2.

(j) $T(f(t)) = f(-t)$ from P_2 to P_2.

(k) $T(f(t)) = f(2t) - f(t)$ from P_2 to P_2.

(3) Define *image*: the image of a linear transformation T is ...

(4) Define *rank*: the rank of a linear transformation T is ...

(5) Define *kernel*: the kernel of a linear transformation T is ...

(6) Define *nullity*: the nullity of a linear transformation T is ...

(7) Find the image, rank, kernel, and nullity of the transformation T from P_2 to P_2 defined by $T(f(t)) = f''(t) + 4f'(t)$.

(8) Find the image, rank, kernel, and nullity of the transformation T from P_2 to $\mathbb{R}$ defined by $T(f(t)) = \int_{-2}^{3} f(t)\,dt$.

(9) Find the image, rank, kernel, and nullity of the transformation T from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ defined by $T(M) = M\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}M$.

SELECTED ANSWERS:

(2) T is a linear transformation from V to W if $T(kf) = kT(f)$ and $T(f+g) = T(f) + T(g)$, for any f and g in V and any scalar k. T is an isomorphism if T is invertible; to show that T is invertible, you either need to find the inverse of T, or show that a matrix for T is invertible.

(a) T is not a linear transformation: $T(M+N) = M+N+I$ and $T(M)+T(N) = M+I+N+I$, thus $T(M+N) \neq T(M)+T(N)$.

(b) T is a linear transformation: $T(M+N) = 7(M+N) = 7M+7N = T(M)+T(N)$; $T(kM) = 7(kM) = k(7M) = kT(M)$.
T is an isomorphism. We can show that T is invertible by writing down the inverse function for T: $T^{-1}(M) = M/7$. To verify that this map is the inverse of T, we check that $T(T^{-1}(M)) = T^{-1}(T(M)) = M$:

$$T(T^{-1}(M)) = T(M/7) = M$$
$$T^{-1}(T(M)) = T^{-1}(7M) = M.$$

We can also show that T is invertible by expressing T as a matrix and showing that that matrix is invertible. To express T as a matrix, we need to fix a basis, and express T in terms of this basis. (See problems 4, 5, and 6 of the next section.) Let's take $\mathfrak{B} = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$. Then the matrix for T in this basis is

$$B = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix},$$

where v_1, v_2, v_3, and v_4 are the basis elements. We compute these columns:

$$T(v_1) = T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 7\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad \text{so} \quad [T(v_1)]_{\mathfrak{B}} = \begin{bmatrix}7\\0\\0\\0\end{bmatrix},$$

$$T(v_2) = T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = 7\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad \text{so} \quad [T(v_1)]_{\mathfrak{B}} = \begin{bmatrix}0\\7\\0\\0\end{bmatrix},$$

$$T(v_3) = T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right) = 7\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \quad \text{so} \quad [T(v_1)]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\7\\0\end{bmatrix},$$

$$T(v_4) = T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = 7\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \quad \text{so} \quad [T(v_1)]_{\mathfrak{B}} = \begin{bmatrix}0\\0\\0\\7\end{bmatrix},$$

hence $B = \begin{bmatrix}7 & 0 & 0 & 0\\0 & 7 & 0 & 0\\0 & 0 & 7 & 0\\0 & 0 & 0 & 7\end{bmatrix}$, which is invertible, so T is invertible.

(c) T is not a linear transformation: $T(kM) = (kM)^2 = k^2M^2$ and $kT(M) = kM^2$, thus $T(kM) \neq kT(M)$.

(h) T is a linear transformation: $T(f(t) + g(t)) = f(7) + g(7) = T(f(t)) + T(g(t))$, and $T(kf(t)) = kf(7) = kT(f(t))$.
T is not a linear transformation. There is no way to write down an inverse function for T; knowing $f(7)$ is not enough to detemine the function $f(t)$.
We can see this by writing down a matrix for T. Let's take the basis $\mathfrak{B} = \{1, t, t^2\}$. Then

$$B = \begin{bmatrix} | & | & | \\ [T(1)]_{\mathfrak{B}} & [T(t)]_{\mathfrak{B}} & [T(t^2)]_{\mathfrak{B}} \\ | & | & | \end{bmatrix},$$

where

$$T(1) = 1 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T(t) = 7 \Rightarrow [T(t)]_{\mathfrak{B}} = \begin{bmatrix}7\\0\\0\end{bmatrix}$$

$$T(t^2) = 49 \Rightarrow [T(t^2)]_{\mathfrak{B}} = \begin{bmatrix}49\\0\\0\end{bmatrix}.$$

So $B = \begin{bmatrix} 1 & 7 & 49 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is not invertible, so T is not invertible.

(k) T is a linear transformation: $T(f(t)+g(t)) = f(2t)+g(2t)-f(t)-g(t) = T(f(t)) + T(g(t))$ and $T(kf(t)) = kf(2t) - kf(t) = k(f(2t)-f(t)) = kT(f(t))$.
T is not an isomorphism. Let's take $\mathfrak{B} = \{1, t, t^2\}$. Then

$$T(1) = 1 - 1 = 0 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $B = \begin{bmatrix} 0 & ? & ? \\ 0 & ? & ? \\ 0 & ? & ? \end{bmatrix}$, which is not invertible, so T is not invertible.

(7) In order to answer these questions, we should find a basis for the image and kernel of the linear transformation. To do this, we should write down a general element in the image and kernel of the map.

Let's find a basis for the image: we have $f(t) = a + bt + ct^2$, and

$$T(f(t)) = T(a + bt + ct^2) = 2c + 4(2ct + b) = c(2 + 8t) + b(4).$$

Hence the image of T is spanned by the polynomials 4 and $8t+2$. These polynomials form a linearly independent set, so we have a basis for the image: $\mathfrak{B} = \{4, 8t+2\}$. There are two elements in the basis, so the rank of T is 2.

By the rank-nullity theorem, $\text{rank}(T) + \text{nullity}(T) = 3$, so the nullity of T is 1. To find the kernel of T, we need a general element in the kernel, that is, we need $T(f(t)) = 0$. From before, we have

$$0 = T(f(t)) = 2c + 4b + 8ct \Rightarrow c = 0 \Rightarrow b = 0$$

Hence $f(t)$ is in the kernel of T if and only if $f(t) = a \cdot 1$, that is, $\ker(T) = \text{span}\{1\}$. Again we see that the nullity of T is one since the dimension of the kernel is 1.

(9) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We have

$$\begin{aligned} T(M) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix} - \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} -2c & 2a-2d \\ 0 & 2c \end{bmatrix} = a\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} + d\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

We see that the image of T is spanned by three matrices, but only two of these are linearly independent; a basis for the image is $\mathfrak{B} = \left\{\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\right\}$, and the rank of this map is 2.

As for the kernel, we need $T(M) = 0$:

$$0 = T(M) = \begin{bmatrix} -2c & 2a-2d \\ 0 & 2c \end{bmatrix},$$

hence $c = 0$, $a = d$, and b is free. So if M is in the kernel of T, then $M = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus a basis for the kernel is $\mathfrak{K} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$, and the nullity of T is 2.

3. Matrix of a linear transformation

(1) If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the *standard matrix* for T in the *standard basis* $\mathfrak{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ of $\mathbb{R}^n$ is

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}.$$

More generally, given any basis $\mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$, the $\mathfrak{B}$-matrix of T is

$$B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}.$$

Moreover, $AS = SB$ for the matrix

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}.$$

These matrices are related according to the diagram

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathfrak{B}}} \arrow[u, "S"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, "S"'] \end{tikzcd}$$

What is the *standard basis* for $\mathbb{R}^n$? That is, what are the vectors $\vec{e}_1, \ldots, \vec{e}_n$?

(2) Let T be the linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix} -9a - 6c \\ -4a + 7b + 2c \\ -4a - 19c \end{bmatrix}.$$

(a) Compute the standard matrix for T.
(b) Let $\vec{x} = \begin{bmatrix} 2 & -1 & -3 \end{bmatrix}^T$. Verify that $T(\vec{x}) = A\vec{x}$.

Let $\mathfrak{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

(c) What is S?
(d) Compute $[\vec{x}]_{\mathfrak{B}}$ and $[T(\vec{x})]_{\mathfrak{B}}$.
(e) Verify that $S[\vec{x}]_{\mathfrak{B}} = \vec{x}$ and $S[T(\vec{x})]_{\mathfrak{B}} = T(\vec{x})$.
(f) Compute B, the $\mathfrak{B}$-matrix of T.
(g) Verify that $B[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}$.
(h) Verify that $AS = SB$.

(3) Repeat problem 2 for the linear transformation T defined by

$$T(\vec{y}) = \vec{y} \times \langle 2, 1, 1 \rangle,$$

the vector $\vec{x} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^T$, and the basis $\mathfrak{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(4) Let V be an n-dimensional vector space, and let T be a linear transformation from V to V. Let $\mathfrak{A} = \{w_1, \ldots, w_n\}$ and $\mathfrak{B} = \{v_1, \ldots, v_n\}$ be two bases for V. Then we have the following diagram:

$$\begin{tikzcd}
{[f]_{\mathfrak{A}}} \arrow[rrr, "A"] & & & {[T(f)]_{\mathfrak{A}}} \\
& f \arrow[ul, "L_{\mathfrak{A}}"'] \arrow[dl, "L_{\mathfrak{B}}"] \arrow[r, "T"] & T(f) \arrow[ur, "L_{\mathfrak{A}}"] \arrow[dr, "L_{\mathfrak{B}}"'] & \\
{[f]_{\mathfrak{B}}} \arrow[uu, "S"] \arrow[rrr, "B"] & & & {[T(f)]_{\mathfrak{B}}} \arrow[uu, "S"']
\end{tikzcd}$$

where

$$A = \begin{bmatrix} \vert & & \vert \\ [T(w_1)]_{\mathfrak{A}} & \cdots & [T(w_n)]_{\mathfrak{A}} \\ \vert & & \vert \end{bmatrix}, \qquad B = \begin{bmatrix} \vert & & \vert \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_n)]_{\mathfrak{B}} \\ \vert & & \vert \end{bmatrix},$$

$$S = \begin{bmatrix} \vert & & \vert \\ [v_1]_{\mathfrak{A}} & \cdots & [v_n]_{\mathfrak{A}} \\ \vert & & \vert \end{bmatrix},$$ and $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$ are the *standard coordinate maps*.

How are these standard coordinate maps defined?

(5) Let T be the linear transformation from P_2 to P_2 defined by

$$T(f(x)) = f(3) + f'(x).$$

Let $\mathfrak{A} = \{1, x, x^2\}$ and $\mathfrak{B} = \{1, (x-3), (x-3)^2\}$.

(a) Compute A, B, and S. Verify that $AS = SB$.

(b) Let $f(x) = 1 + 4x - x^2$. Compute $[f]_{\mathfrak{A}}$, $[f]_{\mathfrak{B}}$, $T(f)$, $[T(f)]_{\mathfrak{A}}$, and $[T(f)]_{\mathfrak{B}}$.

(c) Verify that $A[f]_{\mathfrak{A}} = [T(f)]_{\mathfrak{A}}$, $B[f]_{\mathfrak{B}} = [T(f)]_{\mathfrak{B}}$, $S[f]_{\mathfrak{B}} = [f]_{\mathfrak{A}}$, and $S[T(f)]_{\mathfrak{B}} = [T(f)]_{\mathfrak{A}}$.

(d) Use the matrix B to determine the image, rank, kernel, and nullity of T.

(6) Let W be the set of upper-triangular 2×2 matrices, and let T be the linear transformation from W to W defined by

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Let $\mathfrak{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and $\mathfrak{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$.

(a) Compute A, B, and S. Verify that $AS = SB$.

(b) Let $M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Compute $[M]_{\mathfrak{A}}$, $[M]_{\mathfrak{B}}$, $T(M)$, $[T(M)]_{\mathfrak{A}}$, and $[T(M)]_{\mathfrak{B}}$.

(c) Verify that $A[M]_{\mathfrak{A}} = [T(M)]_{\mathfrak{A}}$, $B[M]_{\mathfrak{B}} = [T(M)]_{\mathfrak{B}}$, $S[M]_{\mathfrak{B}} = [M]_{\mathfrak{A}}$, and $S[T(M)]_{\mathfrak{B}} = [T(M)]_{\mathfrak{A}}$.

(d) Determine the image, rank, kernel, and nullity of T.

(7) Define *isomorphism*: a linear transformation T is an isomorphism if ...

(8) For which constants k is the linear transformation $T(M) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} M - M \begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$ an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$?

(9) Let $\mathbb{R}^+$ be the set of positive real numbers. On this space, addition $\oplus$ and scalar multiplication $\odot$ defined by: $a \oplus b = ab, \quad k \odot a = a^k$.

(a) Show that $\mathbb{R}^+$ with the operations $\oplus$ and $\odot$ is a vector space.

(b) Show that $T(x) = \ln(x)$ is a linear transformation from $\mathbb{R}^+$ to $\mathbb{R}$.

(c) Is T an isomorphism?

SELECTED ANSWERS:

(3) (a) The standard matrix is $A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$, so we compute:

$$T(e_1) = \langle 1,0,0\rangle \times \langle 2,1,1\rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \langle 0,-1,1\rangle = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$T(e_2) = \langle 0,1,0\rangle \times \langle 2,1,1\rangle = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \langle 1,0,-2\rangle = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$T(e_3) = \langle 0,0,1\rangle \times \langle 2,1,1\rangle = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \langle -1,2,0\rangle = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

We have

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

(b) We evaluate:

$$T(\vec{x}) = T(\begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^T) = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}.$$

(c) $S = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(d) To compute $[x]_{\mathfrak{B}}$, we reduce the augmented matrix $\begin{bmatrix} S & \vec{x} \end{bmatrix}$:

$$\begin{bmatrix} 0 & 2 & 0 & 2 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

so $[x]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. To find $[T(x)]_{\mathfrak{B}}$, we reduce $\begin{bmatrix} S & T(x) \end{bmatrix}$:

$$\begin{bmatrix} 0 & 2 & 0 & -4 \\ 1 & -1 & -1 & 4 \\ 1 & -1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

hence $[T(x)]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$.

(e)

$$S[x]_{\mathfrak{B}} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = x$$

$$S[T(x)]_{\mathfrak{B}} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} = T(x).$$

(f) $B = \begin{bmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}$, so we compute:

$$T(v_1) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = -2v_3$$

$$T(v_2) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 4 \end{bmatrix} = 4v_3$$

$$T(v_3) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = v_1 - v_2.$$

Thus $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 0 \end{bmatrix}$.

(g)

$$B[x]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = [T(x)]_{\mathfrak{B}}.$$

(h)

$$AS = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 2 & -4 & 2 \\ -2 & 4 & 2 \end{bmatrix}$$

$$SB = \begin{bmatrix} 0 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 2 & -4 & 2 \\ -2 & 4 & 2 \end{bmatrix}.$$

(5) (a) For A, we have

$$T(1) = 1 \Rightarrow [T(1)]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = 3 + 1 = 4 \Rightarrow [T(x)]_{\mathfrak{A}} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \qquad T(x^2) = 9 + 2x \Rightarrow [T(x^2)]_{\mathfrak{A}} = \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix}.$$

So $A = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. For B, we have

$$T(1) = 1 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x-3) = 0 + 1 = 1 \Rightarrow [T(x-3)]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T((x-3)^2) = 0 + 2(x-3) \Rightarrow [T((x-3)^2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

So $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. For S,

$$[1]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [x-3]_{\mathfrak{A}} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad [(x-3)^2]_{\mathfrak{A}} = [x^2 - 6x + 9]_{\mathfrak{A}} = \begin{bmatrix} 9 \\ -6 \\ 1 \end{bmatrix}$$

So $S = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$. Lastly,

$$AS = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$SB = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) For $[f]_{\mathfrak{A}}$, we just take the coefficients of the polynomial:

$$[f]_{\mathfrak{A}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}.$$

Since $S[f]_{\mathfrak{B}} = [f]_{\mathfrak{A}}$, we can solve for $[f]_{\mathfrak{B}}$ by reducing the matrix $\begin{bmatrix} S & [f]_{\mathfrak{A}} \end{bmatrix}$:

$$\begin{bmatrix} 1 & -3 & 9 & 1 \\ 0 & 1 & -6 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 10 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

hence $[f]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$.

$$T(f) = T(1 + 4x - x^2) = 4 + 4 - 2x = 8 - 2x,$$

$$[T(f)]_{\mathfrak{A}} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix},$$

and $[T(f)]_{\mathfrak{B}}$ comes from reducing $\begin{bmatrix} S & [T(f)]_{\mathfrak{A}} \end{bmatrix}$:

$$\begin{bmatrix} 1 & -3 & 9 & 8 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow [T(f)]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.$$

(c)

$$A[f]_{\mathfrak{A}} = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} = [T(f)]_{\mathfrak{A}}$$

$$B[f]_{\mathfrak{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = [T(f)]_{\mathfrak{B}}$$

$$S[f]_{\mathfrak{B}} = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = [f]_{\mathfrak{A}}$$

$$S[T(f)]_{\mathfrak{B}} = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 0 \end{bmatrix} = [T(f)]_{\mathfrak{A}}$$

(d) B has two pivots, so the rank$(T) = 2$ and nullity$(T) = 1$. The pivot columns of B correspond to the polynomials 1 and $2(x-3)$, thus the $\text{im}(T) = \text{span}\{1, 2(x-3)\}$. In fact, this is a basis for $\text{im}(T)$ as these polynomials are linearly independent. To find the kernel of T, we look at the kernel of B, from which we see that $x_1 = -x_2$ and $x_3 = 0$. Thus $\ker(B) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. This vector corresponds to the polynomial $(x-3)-1$, hence $\ker(T) = \text{span}\{x-4\}$. We verify that $x-4$ is in the kernel of T:

$$T(x-4) = -1 + 1 = 0.$$

4. Eigenspaces

(1) Which of the following matrices A are diagonalizable? If possible, find an invertible matrix S and a diagonal matrix D for which $A = SDS^{-1}$.

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ (g) $\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(2) For what values of a, b, and c are the matrices diagonalizable?

(a) $\begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{bmatrix}$

(3) Find all eigenvalues and "eigenvectors" of the linear transformations and determine if they are diagonalizable.

(a) $T(A) = A + A^T$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(b) $T(A) = A - A^T$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.

(c) $T(x + iy) = x - iy$ from $\mathbb{C}$ to $\mathbb{C}$.

(d) $T(f(x)) = f(-x)$ from P_2 to P_2.

(e) $T(f(x)) = f(3x - 1)$ from P_2 to P_2.

(4) Find all eigenvalues and "eigenvectors" of the linear transformations.
   (a) $T(x_0, x_1, x_2, \ldots) = (x_2, x_3, x_4, \ldots)$ from the space of infinite sequences to itself.
   (b) $T(x_0, x_1, x_2, \ldots) = (x_0, x_2, x_4, \ldots)$.

(5) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, find a basis of the linear space V of all 2×2 matrices S such that $AS = SD$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Find the dimension of V.

5. True/false

T **F** : The space $\mathbb{R}^{2\times 3}$ is 5-dimensional.

T **F** : If $f_1, \ldots, f_n$ is a basis of a linear space V, then any element of V can be written as a linear combination of $f_1, \ldots, f_n$.

T **F** : The space P_1 is isomorphic to $\mathbb{C}$.

T **F** : If the kernel of a linear transformation T from P_4 to P_4 is $\{0\}$ then T is an isomorphism.

T **F** : If T is a linear transformation from P_6 to $\mathbb{R}^{2\times 2}$, then the kernel of T must be 3-dimensional.

T **F** : The polynomials of degree less than 7 form a 7-dimensional vector space.

T **F** : The function $T(f) = 3f - 4f'$ is a linear transformation from C^∞ to C^∞.

T **F** : The kernel of a linear transformation is a subspace of the domain.

T **F** : The linear transformation $T(f) = f + f''$ is an isomorphism from C^∞ to C^∞.

T **F** : All linear transformations from P_3 to $\mathbb{R}^{2\times 2}$ are isomorphisms.

T **F** : If a linear space V can be spanned by 10 elements, then $\dim(V) \leq 10$.

T **F** : If T is an isomorphism, then T^{-1} is an isomorphism.

T **F** : The matrix $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

T **F** : The matrix $\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

6. more

(1) Consider a nonzero 3×3 matrix A such that $A^2 = 0$.
   (a) Show that the image of A is a subspace of the kernel of A.
   (b) Find the dimensions of the image and kernel of A.
   (c) Pick a nonzero vector v_1 in the image of A, and write $v_1 = Av_2$ for some v_2 in $\mathbb{R}^3$. Let v_3 be a vector in the kernel of A that is not a scalar multiple of v_1. Show that $\mathfrak{B} = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.
   (d) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathfrak{B}$.