QUIZ 4
University of Massachusetts Amherst
Math 235 – Spring 2014
February 25, 2014

NAME:

---

(1) (1 point) Give the definition for the *span of a set of vectors*, $\vec{v}_1, \ldots, \vec{v}_m$.

ANSWER: $\text{span}\{\vec{v}_1, \ldots, \vec{v}_m\}$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$.

(2) (1 point) Give the definition for the *kernel of a linear transformation* T.

ANSWER: $\ker(T)$ is the set of all solutions to $T(\vec{x}) = \vec{0}$. It is also the set of all solutions to $A\vec{x} = \vec{0}$, where A is the matrix associated to T.

(3) (3 points) Determine if $\vec{b} = \begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix}$ is in the image of $A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$. If it is, express $\vec{b}$ as a linear combination of the columns of A.

ANSWER: This is asking if there is a solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the equation $A\vec{x} = \vec{b}$. We can find all solutions to this equation by reducing the augmented matrix associated to this system.

$$\begin{bmatrix} 4 & 2 & 0 & -6 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 4 & 2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now that the matrix is in reduced form, we see that $x_1 = -2$, $x_2 = 1$, and $x_3 = 1$ is a solution (in fact, it is the unique solution), so $\vec{b}$ is in $\text{im}(A)$. Recalling that $A\vec{x}$ is a linear combination of the columns of A, we have

$$\vec{b} = A\vec{x} = x_1 \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(4) (1 point each) True/False: circle **T** or **F**.

T **F** : The vector $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is in the kernel of the matrix $\begin{bmatrix} 4 & -1 \\ 8 & 2 \end{bmatrix}$.

False. Recall that a vector $\vec{x}$ is in the kernel if $A\vec{x} = \vec{0}$. To check if $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is in the kernel of the given matrix, we may simply compute the matrix product:

$$\begin{bmatrix} 4 & -1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}.$$

The output is not the zero vector, so $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is not in the kernel.

T **F** : If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\mathrm{im}(A) = \{\vec{0}\}$.

True. For any vector $\vec{x}$, the product $A\vec{x}$ is $\vec{0}$. Hence the only vector in the image is the zero vector.

T **F** : If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\ker(A) = \{\vec{0}\}$.

False. $A\vec{x} = \vec{0}$ for any vector in $\mathbb{R}^2$, so $\ker(A) = \mathbb{R}^2$.

T **F** : If $\vec{x}$ is in the kernel of A, then $\vec{x}$ is in the kernel of A^2. (A is any matrix.)

True. If $\vec{x} \in \ker(A)$, then $A\vec{x} = \vec{0}$. To see if $\vec{x} \in \ker(A^2)$, we compute $A^2\vec{x}$:
$A^2\vec{x} = AA\vec{x} = A(A\vec{x}) = A\vec{0} = \vec{0}$
Hence $\vec{x} \in \ker(A^2)$.

T **F** : If $\vec{b}$ is in the image of A, then $\vec{b}$ is in the image of A^2. Hint: consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

False. The image of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ contains nonzero vectors (for example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$). But $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the image of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is just the zero vector. Hence not every vector in the image of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is in the image of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2$.