2.1 — Linear transformations
University of Massachusetts Amherst
Math 235 — Spring 2014

Definition 1. A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a *linear transformation* if there exists an $n \times m$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$. In particular, T maps vectors of length m to vectors of length n.

Theorem 2. *A transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is linear if and only if*

(i) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$, for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^m$, and

(ii) $T(k\vec{v}) = kT(\vec{v})$, for all vectors $\vec{v}$ in $\mathbb{R}^m$ and all scalars k.

Example 3. Let $\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and T be a linear transformation. Suppose that

$$T(\vec{u}) = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad T(\vec{v}) = \begin{bmatrix} 4 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\vec{w}) = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(i) Evaluate $T(2\vec{v})$.

(ii) Evaluate $T(\vec{v} - \vec{w})$.

(iii) Evaluate $A \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(iv) What are the dimensions of the matrix A associated to T?

(v) Find the matrix A.