# 3.4 — Coordinates
## University of Massachusetts Amherst
## Math 235 — Spring 2014

**Definition 1.** Let $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of a subspace $V$ of $\mathbb{R}^n$. Any vector $\vec{v}$ in $V$ can be written uniquely as a linear combination of the basis vectors:

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

The scalars $a_1, \dots, a_n$ are called the $\mathfrak{B}$-coordinates of $\vec{v}$, and we write

$$[\vec{v}]_{\mathfrak{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

In other words, if $A$ is the matrix whose columns are the basis vectors, then $[\vec{v}]_{\mathfrak{B}}$ is the solution to the equation $A\vec{x} = \vec{v}$

**Theorem 2.** *If $\mathfrak{B}$ is a basis of a subspace $V$ of $\mathbb{R}^n$, then coordinates are linear:*
*(a)* $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}}$
*(b)* $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}}$

**Definition 3.** Consider the linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ and a basis $\mathfrak{B}$ of $\mathbb{R}^n$. The $n \times n$ matrix $B$ that transforms $[\vec{x}]_{\mathfrak{B}}$ to $[T(\vec{x})]_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of $T$:

$$B[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}}.$$

The matrix $B$ is constructed as follows: if the basis vectors are $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, then

$$B = \begin{bmatrix} \\ [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & \cdots & [T(\vec{v}_n)]_{\mathfrak{B}} \\ \\ \end{bmatrix}$$

**Example 4.** Let $L$ be the line in $\mathbb{R}^2$ spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ that projects any vector $\vec{x}$ orthogonally onto the line $L$. Compute $[T(\vec{x})]_{\mathfrak{B}}$ when $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and $\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

**Theorem 5.** *Let $T$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. Let $A$ be the matrix associated to $T$, let $B$ be the $\mathfrak{B}$-matrix for $T$, and let $S$ be the matrix whose columns are the vectors in $\mathfrak{B}$. Then $A = SBS^{-1}$.*

**Definition 6.** Two $n \times n$ matrices $A$ and $B$ are *similar* if there exists an invertible matrix $S$ such that $A = SBS^{-1}$.

**Definition 7.** Let $\mathcal{O}$ be a set of objects. An *equivalence relation* on $\mathcal{O}$ is a relationship between objects in $\mathcal{O}$, denoted by $\sim$, that is

- reflexive: if $a$ is in $\mathcal{O}$, then $a \sim a$,
- symmetric: if $a$ and $b$ are in $\mathcal{O}$ and $a \sim b$, then $b \sim a$,
- transitive: if $a$, $b$, and $c$ are in $\mathcal{O}$ and $a \sim b$ and $b \sim c$, then $a \sim c$.

**Theorem 8.** *Similarity is an equivalence relation. That is,*
*(a) Every $n \times n$ matrix is similar to itself (reflexivity).*
*(b) If $A$ is similar to $B$, then $B$ is similar to $A$ (symmetry).*
*(c) If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$ (transitivity).*