4.1 — Vector spaces
University of Massachusetts Amherst
Math 235 — Spring 2014

Definition 1. A *vector space* (or *linear space*) V is a set with two operations: addition and scalar multiplication. For any f and g in the vector space and for any scalars j and k, these operations satisfy the following rules:

(1) Addition is associative: $(f+g)+h = f+(g+h)$.
(2) Addition is commutative: $f+g = g+h$.
(3) Unique additive identity: "0" that satisfies $f+0=f$.
(4) Unique additive inverse: "$-f$" that satisfies $f+(-f)=0$.
(5) Distributive law for elements of the vector space: $(j+k)f = jf+kf$.
(6) Distributive law for scalar multiplication: $k(f+g) = kf+kg$.
(7) Scalar multiplication is associative: $(jk)f = j(kf)$.
(8) Multiplicative identity: $1f = f$.

Example 2. Examples of vector spaces:

(1) $\mathbb{R}^n$
(2) Functions from $\mathbb{R}$ to $\mathbb{R}$, denoted by $F(\mathbb{R},\mathbb{R})$
  (a) Differentiable functions
  (b) Polynomials
  (c) Polynomials of degree $\leq n$
  (d) Linear equations of n variables
(3) $m \times n$ matrices, denoted by $\mathbb{R}^{m\times n}$
(4) Infinite sequences of real numbers
(5) Complex numbers $\mathbb{C}$: all numbers of the form $a+bi$, where a and b are real numbers, and i is a number that satisfies $i^2=-1$

Definition 3. A subset W of a linear space V is called a *subspace* of V if

(a) W contains the additive identity "0" of V.
(b) W is closed under addition.
(c) W is closed under scalar multiplication.

Example 4. Show that the differentiable functions are a subspace of $F(\mathbb{R},\mathbb{R})$.

Example 5. Show that the set of all polynomials of degree at most 2 are a subspace of $F(\mathbb{R},\mathbb{R})$.

Example 6. Do the matrices that commute with $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ form a subspace of $\mathbb{R}^{2\times 2}$?

Example 7. Does the set of non-invertible 2×2 matrices form a subspace of $\mathbb{R}^{2\times 2}$?

Definition 8. For any vector space V, we have notions of *span*, *linear independence*, *basis*, *dimension**, and *coordinates*. *A vector space may be infinite dimensional.

Example 9. Find a basis for and determine the dimension of $\mathbb{R}^{2\times 2}$.

Example 10. Find a basis for the set of all matrices that commute with $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$.

Example 11. What are the dimensions of the spaces listed in Example 10?