6.1 — Determinants
University of Massachusetts Amherst
Math 235 — Spring 2014

The *determinant* of a square matrix A is a particular number that can be computed from the entries of the matrix. If A is the coefficient matrix for a system of linear equations, then the determinant of A can be used to determine whether the solutions to the system are unique. If A represents a linear transformation T, then the determinant can be used to determine whether T is invertible.

For any square matrix $A = \begin{bmatrix} \quad \end{bmatrix}$, the determinant of A is denoted by $\det(A)$, $\det \begin{bmatrix} \quad \end{bmatrix}$, or $\begin{vmatrix} \quad \end{vmatrix}$.

Example 1. What is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

Definition 2. The (i, j)-*cofactor* of an $n \times n$ matrix A, denoted A_{ij}, is the matrix A with the i-th row and j-th column omitted. The dimension of any cofactor matrix is:

Example 3. Recall that for two vectors $\vec{v} = \langle a, b, c\rangle$ and $\vec{u} = \langle d, e, f\rangle$, their *cross product* is the determinant of the 3×3 matrix

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} =$$

Theorem 4. *The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.*

Example 5. Compute the determinants of the following matrices.

$$\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix} + 1 \begin{vmatrix} 0 & 5 \\ 4 & -3 \end{vmatrix} = 22 - 20 = 2. \text{ (expand down the third column)}$$

$$\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix} = 2 \cdot 5 \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10(7-6) = 10. \text{ (expand across the third row, then the first row)}$$

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{vmatrix} = 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 12. \text{ (expand down the first column, then the second column, then the third row)}$$

Theorem 6. *The determinant of a triangular matrix is the product of elements on its diagonal.*

Example 7. $\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot 3 \cdot (-6) \cdot 1 = -18.$