6.2 — Properties of determinants
University of Massachusetts Amherst
Math 235 — Spring 2014

The effect of row operations on the determinant:

Theorem 1. *Let A be a square matrix.*

1. *If a multiple of one row of A is added to another row to produce the matrix B, then* $\det(B) = \det(A)$.
2. *If two rows of A are swapped to produce B, then* $\det(B) = -\det(A)$.
3. *If a row of A is multiplied by a scalar k to produce B, then* $\det(B) = k\det(A)$.

Example 2. Use row operations to compute

$$\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

ANSWER:

$$\det\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} = 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 2 \cdot 1 \cdot 3 \cdot -3 \cdot 6 = -108.$$

Theorem 3. *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

Proof.

□

Theorem 4. *Let A and B be square matrices.*

1. $\det(I_n) = 1$.
2. $\det(A^T) = \det(A)$
3. *If A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
4. $\det(AB) = \det(A)\det(B)$.
5. *If A and B are similar, then* $\det(A) = \det(B)$.

Example 5. Suppose A and B are square matrices with $\det(A) = -1$ and $\det(B) = 2$. Compute:

$(a)\det(AB)$ $(b)\det(B^5)$ $(c)\det(2A)$ $(d)\det(A^TA)$ $(e)\det(B^{-1}AB)$

ANSWER: a) $\det(AB) = \det(A)\det(B) = -2$.
b) $\det(B^5) = \det(B)^5 = 32$.

c) $\det(2A) = -2^n$ (multiply each row of A by 2).
d) $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 = 1$.
e) $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(A) = -1$.