7.2 — Eigenvalues
University of Massachusetts Amherst
Math 235 — Spring 2014

**Definition 1.** Let $A$ be an $n \times n$ matrix. A nonzero vector $\vec{v}$ is an *eigenvector* of $A$ if

$$A\vec{v} = \lambda\vec{v} \quad \text{for some constant } \lambda.$$

That is, $A\vec{v}$ is a scalar multiple of $\vec{v}$. The constant $\lambda$ is the *eigenvalue* associated to $\vec{v}$.

**Example 2.** Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Is $\vec{u}$ or $\vec{v}$ an eigen vector of $A$?

answer:

$$A\vec{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\vec{u}.$$

So $\vec{u}$ is an eigenvector of $A$ for the eigenvalue $-4$.

$$A\vec{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

is not a multiple of $\vec{v}$, so $\vec{v}$ is not an eigenvector of $A$.

**Definition 3.** The determinant $\det(A - \lambda I)$ is a polynomial in $\lambda$. This polynomial is called the *characteristic equation* of $A$.

**Theorem 4.** *Let $A$ be an $n \times n$ matrix. The number $\lambda$ is an eigenvalue of $A$ if and only if*

$$\det(A - \lambda I) = 0.$$

*In other words, the eigenvalues of $A$ are the roots of its characteristic polynomial.*

**Example 5.** Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix}$.

answer: For $A$:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)(3-\lambda) - 8 \\ &= 3 - 4\lambda + \lambda^2 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1), \end{aligned}$$

hence the eigenvalues of $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ are 5 and $-1$.

For $B$:

$$\det(B - \lambda I) = \det\left(\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$
\begin{aligned}
&= \det\left(\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\
&= \det\begin{bmatrix} 3-\lambda & -2 & 5 \\ 1 & -\lambda & 7 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\
&= (2-\lambda)\det\begin{bmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{bmatrix} \\
&= (2-\lambda)\big(-\lambda(3-\lambda)+2\big) \\
&= (2-\lambda)(\lambda^2-3\lambda+2) \\
&= (2-\lambda)(\lambda-2)(\lambda-1)
\end{aligned}
$$

hence the eigenvalues of $B$ are 2, 2, and 1.

**Theorem 6.** *The eigenvalues of a triangular matrix are the elements on its diagonal.*

**Example 7.** Compute the characteristic polynomial and identify the eigenvalues of the identity matrix $I_n$.

**Example 8.** Compute the characteristic polynomial and identify the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

ANSWER: The characteristic polynomial of this matrix is

$$
\begin{aligned}
\det(A-\lambda I) &= \det\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} 1-\lambda & 2 & 3 & 4 & 5 \\ 0 & 2-\lambda & 3 & 4 & 5 \\ 0 & 0 & 1-\lambda & 2 & 3 \\ 0 & 0 & 0 & 2-\lambda & 3 \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix} \\
&= (1-\lambda)^3(2-\lambda)^2.
\end{aligned}
$$

The eigenvalues are 1, 1, 1, 2, and 2. Using the next definition, we also say that the eigenvalues are 1 (with multiplicity 3) and 2 (with multiplicity 2).

**Definition 9.** The *multiplicity* of an eigenvalue $\lambda_0$ of $A$ is the power to which $(\lambda_0 - \lambda)$ divides the characteristic polynomial of $A$.

**Theorem 10.** *An $n \times n$ matrix has at most $n$ real eigenvalues (counted with multiplicity). If $n$ is odd, then there must be at least 1 real eigenvalue. If $n$ is even, then there may not be any real eigenvalues.*

**Theorem 11.** *Complex eigenvalues come in pairs: if $\lambda = a + bi$ is an eigenvalue of $A$, then $a - bi$ is also an eigenvalue of $A$.*

**Example 12.** What are the all possible arrangements of eigenvalues of $2 \times 2$ matrices?

**Example 13.** Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

ANSWER: The characteristic polynomial of a $2 \times 2$ matrix has degree 2, so it either has two real roots/eigenvalues, or one pair of complex roots/eigenvalues. If the eigenvalues are real, then there is either one unique eigenvalue with multiplicity 2, or two distinct eigenvalues.

**Theorem 14.** *The determinant of a matrix is equal to the product of its eigenvalues. The trace of a matrix is equal to the sum of its eigenvalues. That is, if $A$ is an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, then*

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n,$$
$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 \cdots + \lambda_n.$$

**Example 15.** Verify this theorem for general $2 \times 2$ matrices.

ANSWER: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\operatorname{tr}(A) = a + d$ and $\det(A) = ad - bc$. Moreover, the characteristic polynomial for this matrix is

$$\begin{aligned} \det\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A). \end{aligned}$$

On the other hand, if the eigenvalues of $A$ are $\lambda_1$ and $\lambda_2$, then the characteristic polynomial of $A$ is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2.$$

Combining these two equations, we see that $\operatorname{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1 \lambda_2$.