7.4 — Diagonalization
University of Massachusetts Amherst
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Definition 1. An $n \times n$ matrix A is *diagonalizable* if A is similar to some diagonal matrix D. That is, there exists an $n \times n$ invertible matrix S such that $S^{-1}AS$ is diagonal.

Theorem 2. *Let T be a linear transformation where $T(\vec{x}) = A\vec{x}$ for some $n \times n$ matrix A. Suppose $\mathfrak{D} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an eigenbasis for A, with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then the $\mathfrak{D}$-matrix of T is*

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad \textit{where} \quad S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}.$$

Theorem 3.

(a) A is diagonalizable if and only if there exists an eigenbasis for A.
(b) If A has n distinct eigenvalues, then A is diagonalizable.

Theorem 4. *To determine if an $n \times n$ matrix is A is diagonalizable:*

(a) Find the eigenvalues of A.
(b) For each eigenvalue λ, determine the dimension of the eigenspace E_λ.
(c) The matrix A is diagonalizable if and only if the dimensions of the eigenspaces add up to n.

Example 5. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ if possible.

ANSWER: A is already diagonal, so we don't have to to anything, but let's go through the steps anyway.

First we need to determine the eigenvalues. This is a diagonal matrix, so its eigenvalues are the numbers on the diagonal: 1 (with algebraic multiplicity 1) and 0 (with algebraic multiplicity 2).

Now we need to compute the eigenspaces:

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$E_0 = \ker(A) = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We see that $\dim E_1 = 1$ and $\dim E_0 = 2$. The dimensions add up to 3, so an eigenbasis exists, and the matrix is diagonalizable. Specifically, the eigenbasis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (just take the vectors from the eigenspaces), and the diagonal matrix is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Example 6. For which values is the matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizable?

ANSWER: A is upper triangular, so its eigenvalues are on the diagonal: 0 (with algebraic multiplicity 1) and 1 (with algebraic multiplicity 2). Recall from Theorem 5 of the 7.3 notes that

$$1 \leq \text{geometric multiplicity of } \lambda \leq \text{algebraic multiplicity of } \lambda.$$

The geometric multiplicity of λ is the dimension of the eigenspace E_λ. So for the eigenvalue 0, we know that its geometric multiplicity must be 1. We can verify this by computing the eigenspace:

$$E_0 = \ker(A) = \ker \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} = \ker \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} -a \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For the matrix to be diagonalizable, we need the dimensions of the eigenspaces to add to 3, so we need the dimension of the eigenspace E_1 to be 2. Let's compute this eigenspace.

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & -c \\ 0 & a & b \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & -c \\ 0 & 0 & b+ac \\ 0 & 0 & 0 \end{bmatrix}.$$

The dimension of the kernel is 2 if and only if the matrix has one pivot, and the matrix has exactly one pivot only if $b + ac = 0$. Thus A is diagonalizable if and only if $b + ac = 0$.

Example 7. Let $A = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$. Compute $\lim_{t\to\infty} A^t$.

ANSWER: This question is nearly impossible to do by direct computation. The key is to diagonalize A, that is, write $A = SDS^{-1}$ for some invertible matrix S and diagonal matrix D.

To diagonalize A, we first compute its eigenvalues. The characteristic polynomial of a 2×2 matrix is

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - \frac{3}{4}\lambda - \frac{1}{4}.$$

To find the eigenvalues, we set this polynomial equal to 0:

$$\begin{aligned} \lambda^2 - \frac{3}{4}\lambda - \frac{1}{4} &= 0 \\ 4\lambda^2 - 3\lambda - 1 &= 0 \\ (4\lambda + 1)(\lambda - 1) &= 0, \end{aligned}$$

so the eigenvalues are $-1/4$ and 1. Thus

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}.$$

To get S, we need to compute the eigenspaces:

$$E_1 = \ker(A - I) = \ker \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{3}{4} \end{bmatrix} = \ker \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \right\}$$

$$E_{-1/4} = \ker(A + \frac{1}{4}I) = \ker \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

So let's take $S = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$. Now

$$A = SDS^{-1} = \frac{2}{5} \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}.$$

There are two facts we need to finish this problem. First, $(SDS^{-1})^t = SD^tS^{-1}$ (this property appeared in one of the previous homework assignments). Secondly, $D^t = \begin{bmatrix} 1^t & 0 \\ 0 & (-\frac{1}{4})^t \end{bmatrix}$ (this is a property of diagonal matrices from Chapter 2). Using these properties, we get

$$\lim_{t\to\infty} A^t = \lim_{t\to\infty} SD^tS^{-1}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \frac{2}{5} \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{4}\right)^t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix} \\
&= \frac{2}{5} \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix} \\
&= \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix}.
\end{aligned}
$$

Definition 8. Let V be a vector space, and let T be a linear transformation from V to V. A scalar λ is an eigenvalue for T if there exists an nonzero f in V such that $T(f) = \lambda f$. Such an f is called an *eigenfunction* (or *eigenmatrix*, etc.) if V is a space of functions (or matrices, etc.). In general, we still call f an *eigenvector* though f may not be a vector.

If V is finite dimensional, then a basis $\mathfrak{D}$ of V of eigenvectors of T is an *eigenbasis* of T. T is *diagonalizable* if the matrix for T with respect to some basis is diagonal, and T is diagonalizable if and only if T has an eigenbasis.

Example 9. Consider the map $T(A) = A^T$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$. Is the transformation T diagonalizable? If so, find an eigenbasis for T.

ANSWER: We will do this problem by expressing T as a matrix, and then computing the eigenvalues and eigenvectors of this matrix. We follow the notation from Section 4.3.

First we need to pick a basis for the set of 2×2 matrices. It doesn't matter which basis we choose, so let's take

$$
\mathfrak{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.
$$

The $\mathfrak{B}$-matrix for T is

$$
B = \begin{bmatrix} \\ [T(f_1)]_{\mathfrak{B}} & \cdots & [T(f_n)]_{\mathfrak{B}} \\ \\ \end{bmatrix},
$$

where f_1, f_2, f_3, and f_4 are the matrices in our basis. You may need to review the definition of coordinates from Section 3.4.

Now we compute B:

$$
\begin{aligned}
T(f_1) &= T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
T(f_2) &= T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
T(f_3) &= T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
T(f_4) &= T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}
$$

hence

$$
[T(f_1)]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(f_2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [T(f_3)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(f_4)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and}
$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of T are the eigenvalues of B, so we compute

$$\det(B - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = (\lambda - 1)^3(\lambda + 1).$$

Hence the eigenvalues of B are 1 (with algebraic multiplicity 3) and -1 (with algebraic multiplicity 1).

The eigenspaces are

$$E_1 = \ker \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$E_{-1} = \ker \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The dimensions of the eigenspaces add up to 4, so T is diagonalizable. That is, T is given by the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, and you may verify that $D = S^{-1}BS$ for the matrix $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (the columns of S are the eigenvectors of B).

These four vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ correspond to the eigenmatrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and you may verify that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = -1 \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Example 10. Consider the linear transformation $T(f(x)) = f(2x - 1)$ from P_2 to P_2. Is T diagonalizable? If so, find an eigenbasis $\mathfrak{D}$ and the $\mathfrak{D}$-matrix D of T.

ANSWER: We proceed follow the same steps as in the previous problem, this time with the basis $\mathfrak{B} = \{1, x, x^2\}$. In this basis, the $\mathfrak{B}$-matrix is $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$. B has three distinct eigenvalues, so it is diagonalizable, and the diagonal matrix is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ since the eigenvalues are $1, 2,$ and 4. If you compute the eigenspaces, you will find that

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad E_2 = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad E_4 = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\},$$

which correspond to the eigenfunctions $1, x-1$, and $x^2 - 2x + 1$. We verify that these are indeed eigenfunctions for their corresponding eigenvalues.

$$T(1) = 1 = 1 \cdot 1.$$
$$T(x-1) = (2x-1) - 1 = 2 \cdot (x-1).$$
$$T(x^2 - 2x + 1) = (2x-1)^2 - 2(2x-1) + 1 = 4x^2 - 8x + 4 = 4 \cdot (x^2 - 2x + 1).$$

Example 11. Let V be the space of all infinite sequences of real numbers, and let T be the shift map:

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

Find all eigenvalues and eigensequences of T.

ANSWER: A sequence $(x_0, x_1, x_2, \ldots)$ is an eigensequence for the eigenvalue λ if and only if

$$T(x_0, x_1, x_2, \ldots) = \lambda(x_0, x_1, x_2, \ldots).$$

On the other hand, by the definition of T, we have

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

Thus to be an eigensequence, we have

$$\lambda(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots),$$

which implies that

$$\begin{aligned} x_1 &= \lambda x_0 \\ x_2 &= \lambda x_1 = \lambda^2 x_0 \\ x_3 &= \lambda x_2 = \lambda^3 x_0 \\ &\vdots \end{aligned}$$

hence an eigensequnce has the form $(x_0, \lambda x_0, \lambda^2 x_0, \ldots)$. Moreover, we are free to choose λ and x_0. So, every real number is an eigenvalue for T, and for each eigenvalue λ, the corresponding eigenspace is

$$E_\lambda = \text{span}\{(1, \lambda, \lambda^2, \ldots)\}.$$