7.5 — Complex eigenvalues
University of Massachusetts Amherst
Math 235 — Spring 2014

Theorem 1. *If A is a real 2×2 matrix with eigenvalues $a \pm bi$ ($b \neq 0$), and if $\vec{v} + i\vec{w}$ is an eigenvector of A corresponding to the eigenvalue $a + bi$, then*

$$C^{-1}AC = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \text{where} \quad C = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$

The matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *is known as a* rotation-scaling matrix.

Example 2. Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$.

(a) Diagonalize A: Find a diagonal matrix D and an invertible matrix S such that $AS = SD$.

(b) Compute the rotation-scaling matrix for A: Find an invertible matrix C such that $AC = C \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

answer: We need to compute the eigenvalues and eigenvectors of A.

Computing the eigenvalues:

$$\det(A - \lambda I) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2;$$

$$\lambda = \frac{2 \pm \sqrt{4 - 2 \cdot 4}}{2} = 1 \pm i, \quad \text{by the quadratic formula.}$$

Computing the eigenspaces:

$$E_{1+i} = \ker(A - (1+i)I) = \ker \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 5 \\ 2-i \end{bmatrix} \right\}$$

$$E_{1-i} = \ker(A - (1-i)I) = \ker \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 5 \\ 2+i \end{bmatrix} \right\}.$$

a) We have $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$, and $S = \begin{bmatrix} 5 & 5 \\ 2-i & 2+i \end{bmatrix}$. We verify that $AS = SD$:

$$AS = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 2-i & 2+i \end{bmatrix} = \begin{bmatrix} 5+5i & 5-5i \\ 3+i & 3-i \end{bmatrix}$$

$$SD = \begin{bmatrix} 5 & 5 \\ 2-i & 2+i \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} = \begin{bmatrix} 5+5i & 5-5i \\ 3+i & 3-i \end{bmatrix}.$$

b) Taking the eigenvalue $1+i$ and a corresponding eigenvector $\begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i$, we have $C = \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. We verify that $AC = C \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$:

$$AC = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 1 & 3 \end{bmatrix}$$

$$C \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 1 & 3 \end{bmatrix}.$$