The Problem with Rationals. (Due Friday; Do with a Partner.)

In class we will focus on defining the real numbers. But there is a prior question of how to define the rational numbers. In earlier courses, you defined a rational number as any number that can be expressed as the ratio \( p/q \) of two integers, with the denominator \( q \) not equal to zero. The set of all rational numbers is usually denoted by \( \mathbb{Q} \); it was so named in 1895 by Peano after *quoziente*, Italian for "quotient".

The problem with this definition is that it already presumes that there is some larger set of numbers (the real numbers) of which the rationals are a subset. This is apparent when we say "as any number that can be expressed. . . ." We are thinking that the rationals already exist and are describing them as a subclass of some larger set. This is even more obvious when we write out the formal definition:

\[
\mathbb{Q} = \left\{ r \in \mathbb{R} : r = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.
\]

Clearly, the rational number \( r \) is already thought of as a pre-existing real number.

But the reals are not yet available to us. How might we define the rationals from first principles? It is not enough to start with the integers and say that \( r = \frac{p}{q} \), where \( p, q \in \mathbb{Z} \) and \( q \neq 0 \). The expression \( \frac{p}{q} \) may have no meaning within the integers, e.g., \( \frac{3}{2} \) is not an integer. Using Math 204 language, \( \mathbb{Z} \) is not closed under division (but \( \mathbb{Z} \) is closed under addition, subtraction, and multiplication).

1.1 The Rationals as Ordered Pairs

Recall that the set \( \mathbb{Z}^* = \mathbb{Z} \setminus \{0\} \) is the set of non-zero integers. The rational numbers can be formally defined as the equivalence classes of the set \( (\mathbb{Z} \times \mathbb{Z}^*, \sim) \), where the cartesian product \( \mathbb{Z} \times \mathbb{Z}^* \) is the set of all ordered pairs \((m, n)\) where \( m \) and \( n \) are integers and \( n \neq 0 \), and "\( \sim \)" is the equivalence relation defined by \((m_1, n_1) \sim (m_2, n_2)\) if and only if \( m_1n_2 - m_2n_1 = 0 \).

**Example 1.1.** \((4, 7) \sim (−12, −21)\) because \(4 \times (−21) - 7 \times (−12) = 0\). Similarly, \((2, 5) \not\sim (3, 10)\) because \(2 \times 10 - 5 \times 3 \neq 0\).

**You Try It 1.1.** Prove that \( \sim \) is an equivalence relation on \( \mathbb{Z} \times \mathbb{Z}^* \). That is, prove that \( \sim \) is reflexive, symmetric, and transitive. You may wish to look back at your Math 135 notes and Chapter 4 in the text *Chapter Zero*. Avoid using fractions or division by integers in your proof.

You should only need to use expressions that contain integers.

1.2 Operations on the Rationals

The ordered pair \((p, q)\) represents the rational number that we ordinarily denote as \(\frac{p}{q}\). Notice that \(\frac{p_1}{q_1} = \frac{p_2}{q_2}\) if and only if \(p_1q_2 = p_2q_1\) or \(p_1q_2 - p_2q_1 = 0\) that is true if and only if \((p_1, q_1) \sim (p_2, q_2)\). That is, two ordinary factions are equal exactly when their corresponding ordered pairs are equivalent.

We define **addition** of two ordered pairs \((a, b)\) and \((c, d)\) differently than we define addition of two vectors. To emphasize the difference let’s use the symbol \(\oplus\). Define

\[
(a, b) \oplus (c, d) = (ad + bc, bd).
\]

We need to make sure that when we add equivalent ordered pairs we get equivalent answers. For example, since \((9, 6) \sim (3, 2)\) and \((2, 5) \sim (4, 10)\), check that \((9, 6) \oplus (2, 5)\) is equivalent to \((3, 2) \oplus (4, 10)\).
This class corresponds to the rational language of ordered pairs. The representation of a rational as \( p \) talk about rational numbers without resorting to "ratios of integers" by using the symbolic shorthand for the ordered pair \((p, q)\). Since there are multiple ways to represent the same rational, \( \frac{1}{2} = \frac{2}{4} \), etc., we define an equivalence relation on the ordered pairs so that two rationals are equal precisely when the two ordered pairs are equivalent. We then show that the basic arithmetic operations of addition, multiplication, subtraction, division work as expected on equivalence classes of ordered pairs in \( \mathbb{Z} \times \mathbb{Z}^* \). You have checked some of those details above. But there are others still to check such as commutativity or distributivity.

\[
[p, q] = \{(a, b) \in \mathbb{Z} \times \mathbb{Z}^* : (a, b) \sim (p, q)\}.
\]

If we want to add two rational numbers, say \( \frac{a}{q_1} + \frac{b}{q_2} \), we actually add the two equivalence classes \([p_1, q_1] + [p_2, q_2]\) by adding any two elements in the corresponding equivalence classes, say \((a_1, b_1) \oplus (a_2, b_2) = (a_1b_2 + a_2b_1, b_1b_2)\) and taking the equivalence class of the sum, \([a_1b_2 + a_2b_1, b_1b_2]\). So

\[
[p_1, q_1] + [p_2, q_2] = [a_1b_2 + a_2b_1, b_1b_2].
\]

The exercises above show that no matter which ordered pairs in the equivalence classes for \( \frac{a}{q_1} \) and \( \frac{b}{q_2} \) you choose, you get the same answer.

**Example 1.2.1.** Suppose we want to add \( \frac{4}{11} \) and \( \frac{5}{7} \). Well, this corresponds to adding the classes \([6, 8] + [5, 3]\). We do this addition of classes by adding any elements in the corresponding classes of \([6, 8] + [5, 3]\), say3

\[
(3, 4) \oplus (-15, -9) = (3(-9) + (-15)4, 4(-9)) = (-87, -36).
\]

Now we take the class of this sum to get the answer:

\[
[6, 8] + [5, 3] = [-87, -36].
\]

This class corresponds to the rational \( -\frac{29}{12} \) which is the same as \( \frac{29}{12} \). That is, \((-87, -36) \sim (29, 12)\). Notice that

\[
\frac{6}{8} + \frac{5}{3} = \frac{18 + 40}{24} = \frac{29}{12}.
\]

Though adding two equivalence classes is a pain, everything works out as expected.

**Take-home message** Even if we knew nothing about real numbers, we could still talk about rational numbers without resorting to "ratios of integers" by using the language of ordered pairs. The representation of a rational as \( \frac{p}{q} \) is just a convenient symbolic shorthand for the ordered pair \((p, q)\). Since there are multiple ways to represent the same rational, \( \frac{1}{2} = \frac{2}{4} \), etc., we define an equivalence relation on the ordered pairs so that two rationals are equal precisely when the two ordered pairs are equivalent. We then show that the basic arithmetic operations of addition, multiplication, subtraction, division work as expected on equivalence classes of ordered pairs in \( \mathbb{Z} \times \mathbb{Z}^* \). You have checked some of those details above. But there are others still to check such as commutativity or distributivity.

**YOU TRY IT 1.2.** More generally, prove: If \((m_1, n_1) \sim (m_2, n_2)\) and \((p_1, q_1) \sim (p_2, q_2)\), then

\[
(m_1, n_1) \oplus (p_1, q_1) \sim (m_2, n_2) \oplus (p_2, q_2).
\]

**YOU TRY IT 1.3.** We define multiplication of two ordered pairs \((a, b)\) and \((c, d)\) in \( \mathbb{Z} \times \mathbb{Z}^* \) by

\[
(a, b) \otimes (c, d) = (ac, bd).
\]

So, we need to make sure that when we multiply equivalent ordered pairs we get equivalent answers. Prove that if \((m_1, n_1) \sim (m_2, n_2)\) and \((p_1, q_1) \sim (p_2, q_2)\), then

\[
(m_1, n_1) \otimes (p_1, q_1) \sim (m_2, n_2) \otimes (p_2, q_2).
\]

What about subtraction and division? Well, it is useful to define additive and multiplicative inverses first. If \((a, b) \in \mathbb{Z} \times \mathbb{Z}^*\), then its additive inverse is \((-a, b)\). Similarly, if If \((a, b) \in \mathbb{Z} \times \mathbb{Z}^*\) and \(a \neq 0\), then its multiplicative inverse is \((b, a)\).

If \(a, b \in \mathbb{Z} \times \mathbb{Z}^*\), then its multiplicative inverse is \((b, a)\).

\[
\frac{a}{b} = \frac{b}{a}.
\]

It is easy to check that equivalent ordered pairs have equivalent inverses: If \((a, b) \sim (c, d)\), then \((-a, b) \sim (-c, d)\); If \((a, b) \sim (c, d)\) and \(a \neq 0\) and \(c \neq 0\), then \((b, a) \sim (d, c)\).

\[
\text{More simply, we could have added } (6, 8) \oplus (5, 3). \text{ Try this and check that you get the same answer.}
\]