The Problem with Rationals. (Due Friday; Do with a Partner.)

In class we will focus on defining the real numbers. But there is a prior question of how to define the rational numbers. In earlier courses, you defined a **rational number** as any number that can be expressed as the ratio p/q of two integers, with the denominator q not equal to zero. The set of all rational numbers is usually denoted by $\mathbb{Q}$; it was so named in 1895 by Peano after *quoziente*, Italian for "quotient".

The problem with this definition is that it already presumes that there is some larger set of numbers (the real numbers) of which the rationals are a subset. This is apparent when we say "as any *number* that can be expressed. . . ." We are thinking that the rationals already exist and are describing them as a subclass of some larger set. This is even more obvious when we write out the formal definition:

$$\mathbb{Q} = \left\{ r \in \mathbb{R} : r = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}.$$

Clearly, the rational number r is already thought of as a pre-existing real number.

But the reals are not yet available to us. How might we define the rationals from first principles? It is not enough to start with the integers and say that $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. The expression $\frac{p}{q}$ may have no meaning within the integers, e.g., $\frac{3}{2}$ is not an integer. Using Math 204 language, $\mathbb{Z}$ *is not closed under division* (but $\mathbb{Z}$ is closed under addition, subtraction, and multiplication).

1.1 The Rationals as Ordered Pairs

Recall that the set $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ is the set of non-zero integers. The rational numbers can be formally defined as the equivalence classes of the set $(\mathbb{Z} \times \mathbb{Z}^*, \sim)$, where the cartesian product $\mathbb{Z} \times \mathbb{Z}^*$ is the set of all ordered pairs (m, n) where m and n are integers and $n \neq 0$, and "$\sim$" is the equivalence relation defined by $(m_1, n_1) \sim (m_2, n_2)$ if and only if $m_1 n_2 - m_2 n_1 = 0$.

EXAMPLE 1.1.1. $(4, 7) \sim (-12, -21)$ because $4 \times (-21) - 7 \times (-12) = 0$. Similarly, $(2, 5) \not\sim (3, 10)$ because $2 \times 10 - 5 \times 3 \neq 0$.

YOU TRY IT 1.1. Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$. That is, prove that $\sim$ is reflexive, symmetric, and transitive. You may wish to look back at your Math 135 notes and Chapter 4 in the text *Chapter Zero*. *Avoid using fractions or division by integers in your proof.* You should only need to use expressions that contain integers.

1.2 Operations on the Rationals

The ordered pair (p, q) represents the rational number that we ordinarily denote as $\frac{p}{q}$. Notice that $\frac{p_1}{q_1} = \frac{p_1}{q_1}$ if and only if $p_1 q_2 = p_2 q_1$ or $p_1 q_2 - p_2 q_1 = 0$ that is true if and only if $(p_1, q_1) \sim (p_2, q_2)$. That is, two ordinary factions are equal exactly when their corresponding ordered pairs are equivalent.

We define **addition** of two ordered pairs (a, b) and (c, d) differently than we define addition of two vectors. To emphasize the difference let's use the symbol $\oplus$. Define

$$(a, b) \oplus (c, d) = (ad + bc, bd).$$

We need to make sure that when we add equivalent ordered pairs we get equivalent answers. For example, since $(9, 6) \sim (3, 2)$ and $(2, 5) \sim (4, 10)$, check that $(9, 6) \oplus (2, 5)$ is equivalent to $(3, 2) \oplus (4, 10)$.

YOU TRY IT 1.2. More generally, prove: If $(m_1, n_1) \sim (m_2, n_2)$ and $(p_1, q_1) \sim (p_2, q_2)$, then

$$(m_1, n_1) \oplus (p_1, q_1) \sim (m_2, n_2) \oplus (p_2, q_2).$$

☞ You might find it easier to do the multiplication problem first.

YOU TRY IT 1.3. We define multiplication of two ordered pairs (a, b) and (c, d) in $\mathbb{Z} \times \mathbb{Z}^*$ by

$$(a, b) \otimes (c, d) = (ac, bd).$$

So, we need to make sure that when we multiply equivalent ordered pairs we get equivalent answers. Prove that if $(m_1, n_1) \sim (m_2, n_2)$ and $(p_1, q_1) \sim (p_2, q_2)$, then

$$(m_1, n_1) \otimes (p_1, q_1) \sim (m_2, n_2) \otimes (p_2, q_2).$$

What about subtraction and division? Well, it is useful to define additive and multiplicative inverses first. If $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$, then its **additive inverse** is $(-a, b)$. Similarly, if If $(a, b) \in \mathbb{Z} \times \mathbb{Z}^*$ and $a \neq 0$, then its **multiplicative inverse** is (b, a).

It is easy to check that equivalent ordered pairs have equivalent inverses: If $(a, b) \sim (c, d)$, then $(-a, b) \sim (-c, d)$; If $(a, b) \sim (c, d)$ and $a \neq 0$ and $c \neq 0$, then $(b, a) \sim (d, c)$.

What does this show?

From Math 135 you know that an equivalence relation on a set creates a partition of the set into disjoint *equivalence classes*. What we have shown is that each rational number $\frac{p}{q}$ can be thought of as an equivalence class of ordered pairs. We denote this equivalence class by $[p, q]$, where

$$[p, q] = \{(a, b) \in \mathbb{Z} \times \mathbb{Z}^* : (a, b) \sim (p, q)\}.$$

If we want to add two rational numbers, say $\frac{p_1}{q_1} + \frac{p_2}{q_2}$ we actually add the two equivalence classes $[p_1, q_1] + [p_2, q_2]$ by adding any two elements in the corresponding equivalence classes, say $(a_1, b_1) \oplus (a_2, b_2) = (a_1 b_2 + a_2 b_1, b_1 b_2)$ and taking the equivalence class of the sum, $[a_1 b_2 + a_2 b_1, b_1 b_2]$. So

$$[p_1, q_1] + [p_2, q_2] = [a_1 b_2 + a_2 b_1, b_1 b_2].$$

The exercises above show that no matter which ordered pairs in the equivalence classes for $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ you choose, you get the same answer.

EXAMPLE 1.2.1. Suppose we want to add $\frac{6}{8}$ and $\frac{5}{3}$. Well, this corresponds to adding the classes $[6, 8] + [5, 3]$. We do this addition of classes by adding any elements in the corresponding classes of $[6, 8] + [5, 3]$, say[1]

$$(3, 4) \oplus (-15, -9) = (3(-9) + (-15)4, 4(-9)) = (-87, -36).$$

Now we take the class of this sum to get the answer:

$$[6, 8] + [5, 3] = [-87, -36].$$

This class corresponds to the rational $\frac{-87}{-36}$ which is the same as $\frac{29}{12}$. That is, $(-87, -36) \sim (29, 12)$. Notice that

$$\frac{6}{8} + \frac{5}{3} = \frac{18 + 40}{24} = \frac{29}{12}.$$

Though adding two equivalence classes is a pain, everything works out as expected.

[1] More simply, we could have added $(6, 8) \oplus (5, 3)$. Try this and check that you get the same answer.

Take-home message Even if we knew nothing about real numbers, we could still talk about rational numbers without resorting to "ratios of integers" by using the language of ordered pairs. The representation of a rational as $\frac{p}{q}$ is just a convenient symbolic shorthand for the ordered pair (p, q). Since there are multiple ways to represent the same rational, $\frac{1}{2} = \frac{2}{4}$, etc., we define an equivalence relation on the ordered pairs so that two rationals are equal precisely when the two ordered pairs are equivalent. We then show that the basic arithmetic operations of addition, multiplication, subtraction, division work as expected on equivalence classes of ordered pairs in $\mathbb{Z} \times \mathbb{Z}^*$. You have checked some of those details above. But there are others still to check such as commutativity or distributivity.