Proof

for all integers n, if n is even, then n^2 is even

formal proof – write as an argument and explicitly state each step

n is even
$\therefore\ n^2$ is even

Let n be an arbitrary integer.

1.	n is even	premise
2.	if n is even, then $n = 2k$ for some integer k	definition of even
3.	$n = 2k$ for some integer k	from 1, 2 (*modus ponens*)
4.	if $n = 2k$ for some integer k, then $n^2 = 4k^2$ for that integer k	basic algebra
5.	$n^2 = 4k^2$ for some integer k	from 3, 4 (*modus ponens*)
6.	if $n^2 = 4k^2$ for some integer k, then $n^2 = 2(2k^2)$ for that k	basic algebra
7.	$n^2 = 2(2k^2)$ for some integer k	from 5, 6 (*modus ponens*)
8.	if $n^2 = 2(2k^2)$ for some integer k, then $n^2 = 2k'$ for some integer k'	basic fact about integers
9.	$n^2 = 2k'$ for some integer k'	from 7, 8 (*modus ponens*)
10.	if $n^2 = 2k'$ for some integer k', then n^2 is even	definition of even
11.	n^2 is even	from 9, 10 (*modus ponens*)

less formal – leave out explicit implications and instances of *modus ponens*

Proof. Let n be an arbitrary integer.

1.	n is even	premise
2.	$n = 2k$ for some integer k	definition of even
3.	$n^2 = 4k^2$ for that integer k	basic algebra
4.	$n^2 = 2(2k^2)$ for that k	basic algebra
5.	$n^2 = 2k'$ for some integer k'	substituting $k' = 2k^2$
6.	n^2 is even	definition of even

Since n was an arbitrary integer, the statement is true for all integers. □

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3.	$n^2 = 4k^2$ for that integer k	basic algebra
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5.	$n^2 = 2k'$ for some integer k'	substituting $k' = 2k^2$
6.	n^2 is even	definition of even

Since n was an arbitrary integer, the statement is true for all integers. □

typical "formal" proof – written in prose, may not explicitly state all rules if reader can be expected to fill in those details

Proof. Let n be an arbitrary integer and assume n is even. Then $n = 2k$ for some integer k by the definition of even, and $n^2 = 4k^2 = 2(2k^2)$. Since the product of integers is an integer, we have $n^2 = 2k'$ for some integer k'. Therefore n^2 is even. Since n was an arbitrary integer, the statement is true for all integers. □

Discovering Proofs – Advice

- use the definition
  - e.g. a proof about even numbers likely needs to utilize the definition of "even"
- use the hypotheses
  - the *hypotheses* are the assumptions on which the conclusion will be based
  - previously-proved results can be used

Discovering Proofs – Patterns

- $\forall x\ P(x)$
  - tactic: let x be an arbitrary entity in the domain of discourse; show $P(x)$ is true
    - showing $P(x)$ for any entity means that it must hold for all entities
    - don't use any facts about x beyond what is assumed
- $\exists x\ P(x)$ – *existence proof*
  - tactic: find an example – a specific entity a for which $P(a)$ is true
- disproving things
  - disprove $\forall x\ P(x)$ with an example showing $\exists x\ \neg P(x)$ – a *counterexample*
  - disprove $\exists x\ P(x)$ by showing $\forall x\ \neg P(x)$

Discovering Proofs – Patterns

- $p \rightarrow q$ aka $\forall x\ (P(x) \rightarrow Q(x))$
  - tactic: assume p, show q
    - $p \rightarrow q$ is true when p is false, so only need to handle the case where p is true
  - tactic: show a chain of valid implications $p \rightarrow r \rightarrow s \rightarrow \ldots \rightarrow q$
  - tactic: show the contrapositive ($\neg q \rightarrow \neg p$) – *indirect proof*
- $p \wedge q$
  - tactic: show p and q separately
- $p \vee q$
  - tactic: assume $\neg p$ and show q (based on $p \vee q \equiv \neg p \rightarrow q$)
- $p \leftrightarrow q$
  - tactic: show $p \rightarrow q$ and $q \rightarrow p$
    - in English: p if and only if q, p is necessary and sufficient for q
  - tactic: show a chain of valid biconditionals $p \leftrightarrow r \leftrightarrow s \leftrightarrow \ldots \leftrightarrow q$

4. Determine whether each of the following statements is true. If it true, prove it. If it is false, give a counterexample.
  a) Every prime number is odd.
  b) Every prime number greater than 2 is odd.
  c) If x and y are integers with $x < y$, then there is an integer z such that $x < z < y$.
  d) If x and y are real numbers with $x < y$, then there is a real number z such that $x < z < y$.

5. Suppose that r, s, and t are integers, such that r evenly divides s and s evenly divides t. Prove that r evenly divides t.

6. Prove that for all integers n, if n is odd then n^2 is odd.

7. Prove that an integer n is divisible by 3 iff n^2 is divisible by 3. (Hint: give an indirect proof of "if n^2 is divisible by 3 then n is divisible by 3.")

8. Prove or disprove each of the following statements.
  a) The product of two even integers is even.
  b) The product of two integers is even only if both integers are even.
  c) The product of two rational numbers is rational.
  d) The product of two irrational numbers is irrational.
  e) For all integers n, if n is divisible by 4 then n^2 is divisible by 4.
  f) For all integers n, if n^2 is divisible by 4 then n is divisible by 4.