# Proof by Contradiction

If we start with a set of assumptions and apply rules of logic to derive a statement that is known to be false, then at least one of the assumptions must be false.

- *proof by contradiction*
  - to show *p*, assume ¬*p*
  - seek to derive a statement known to be false, typically both showing that both *q* and ¬*q* follow from the assumption ¬*p*



- The ***natural numbers*** (denoted $\mathbb{N}$) are the numbers $0, 1, 2, \ldots$. Note that the sum and product of natural numbers are natural numbers.
- The ***integers*** (denoted $\mathbb{Z}$) are the numbers $0, -1, 1, -2, 2, -3, 3, \ldots$. Note that the sum, product, and difference of integers are integers.
- The ***rational numbers*** (denoted $\mathbb{Q}$) are all numbers that can be written in the form $\frac{m}{n}$ where $m$ and $n$ are integers and $n \neq 0$. So $\frac{1}{3}$ and $\frac{-65}{7}$ are rationals; so, less obviously, are 6 and $\frac{\sqrt{27}}{\sqrt{12}}$ since $6 = \frac{6}{1}$ (or, for that matter, $6 = \frac{-12}{-2}$), and $\frac{\sqrt{27}}{\sqrt{12}} = \sqrt{\frac{27}{12}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$. Note the restriction that the number in the denominator cannot be 0: $\frac{3}{0}$ is not a number at all, rational or otherwise; it is an undefined quantity. Note also that the sum, product, difference, and quotient of rational numbers are rational numbers (provided you don't attempt to divide by 0.)
- The ***real numbers*** (denoted $\mathbb{R}$) are numbers that can be written in decimal form, possibly with an infinite number of digits after the decimal point. Note that the sum, product, difference, and quotient of real numbers are real numbers (provided you don't attempt to divide by 0.)
- The ***irrational numbers*** are real numbers that are not rational, i.e. that cannot be written as a ratio of integers. Such numbers include $\sqrt{3}$ (which we will prove is not rational) and $\pi$ (if anyone ever told you that $\pi = \frac{22}{7}$, they lied—$\frac{22}{7}$ is only an *approximation* of the value of $\pi$).



- An integer $n$ is ***divisible by*** $m$ iff $n = mk$ for some integer $k$. (This can also be expressed by saying that $m$ **evenly** divides $n$.) So for example, $n$ is divisible by 2 iff $n = 2k$ for some integer $k$; $n$ is divisible by 3 iff $n = 3k$ for some integer $k$, and so on. Note that if $n$ is *not* divisible by 2, then $n$ must be 1 more than a multiple of 2 so $n = 2k+1$ for some integer $k$. Similarly, if $n$ is not divisible by 3 then $n$ must be 1 or 2 more than a multiple of 3, so $n = 2k + 1$ or $n = 2k + 2$ for some integer $k$.

- An integer is ***even*** iff it is divisible by 2 and ***odd*** iff it is not.
- An integer $n > 1$ is ***prime*** if it is divisible by exactly two positive integers, namely 1 and itself. Note that a number must be greater than 1 to even have a chance of being termed "prime". In particular, neither 0 nor 1 is prime.



**1.** Suppose that $a_1, a_2, \ldots, a_{10}$ are real numbers, and suppose that $a_1 + a_2 + \cdots + a_{10} > 100$. Use a proof by contradiction to conclude that at least one of the numbers $a_i$ must be greater than 10.

**3.** The ***pigeonhole principle*** is the following obvious observation: If you have $n$ pigeons in $k$ pigeonholes and if $n > k$, then there is at least one pigeonhole that contains more than one pigeon. Even though this observation seems obvious, it's a good idea to prove it. Prove the pigeonhole principle using a proof by contradiction.

**2.** Prove that each of the following statements is true. In each case, use a proof by contradiction. Remember that the negation of $p \to q$ is $p \wedge \neg q$.

**a)** Let $n$ be an integer. If $n^2$ is an even integer, then $n$ is an even integer.
**b)** $\sqrt{2}$ is irrational.
**c)** If $r$ is a rational number and $x$ is an irrational number, then $r + x$ is an irrational number. (That is, the sum of a rational number and an irrational number is irrational.)
**d)** If $r$ is a non-zero rational number and $x$ is an irrational number, then $rx$ is an irrational number.
**e)** If $r$ and $r + x$ are both rational, then $x$ is rational.