Notation

Notation	Definition
$a \in A$	a is a member (or element) of A
$a \notin A$	$\neg(a \in A)$, a is not a member of A
$\emptyset$	the empty set, which contains no elements
$A \subseteq B$	A is a subset of B, $\forall x(x \in A \rightarrow x \in B)$
$A \subsetneq B$	A is a proper subset of B, $A \subseteq B \wedge A \neq B$
$A \supseteq B$	A is a superset of B, same as $B \subseteq A$
$A \supsetneq B$	A is a proper superset of B, same as $B \supsetneq A$
$A = B$	A and B have the same members, $A \subseteq B \wedge B \subseteq A$
$A \cup B$	union of A and B, $\{x \mid x \in A \vee x \in B\}$
$A \cap B$	intersection of A and B, $\{x \mid x \in A \wedge x \in B\}$
$A \smallsetminus B$	set difference of A and B, $\{x \mid x \in A \wedge x \notin B\}$
$\mathcal{P}(A)$	power set of A, $\{X \mid X \subseteq A\}$

Figure 2.1: Some of the notations that are defined in this section. A and B are sets, and a is an entity.

also { }

$(\forall x \in A)(P(x))$

is true iff P(a) for every element *a* of the set A

$(\exists x \in A)(P(x))$

is true iff there is some element *a* of the set A for which P(a) is true

10. Suppose that A, B, and C are sets. Show that $C \subseteq A \cap B$ if and only if $(C \subseteq A) \wedge (C \subseteq B)$.

11. Suppose that A, B, and C are sets, and that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

12. Suppose that A and B are sets such that $A \subseteq B$. Is it necessarily true that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$? Why or why not?

Boolean Algebra for Sets

- many of the rules of logic have analogs in set theory
  - one can blur the distinction between a predicate and the set of elements for which that predicate is true

Double complement	$\overline{\overline{A}} = A$
Miscellaneous laws	$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$ $\emptyset \cup A = A$ $\emptyset \cap A = \emptyset$
Idempotent laws	$A \cap A = A$ $A \cup A = A$
Commutative laws	$A \cap B = B \cap A$ $A \cup B = B \cup A$
Associative laws	$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Logic	Set Theory
$\mathbb{T}$	U
$\mathbb{F}$	$\emptyset$
$p \wedge q$	$A \cap B$
$p \vee q$	$A \cup B$
$\neg p$	$\overline{A}$

let *U* be a universal set and $A \subseteq U$

the *complement* of *A* in *U* is $\overline{A} = \{ x \in U \mid x \notin A \}$

1. Use the laws of logic to verify the associative laws for union and intersection. That is, show that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.

2. Show that for any sets A and B, $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

3. Recall that the symbol $\oplus$ denotes the logical exclusive or operation. If A and B sets, define the set $A \mathbin{\triangle} B$ by $A \mathbin{\triangle} B = \{x \mid (x \in A) \oplus (x \in B)\}$. Show that $A \mathbin{\triangle} B = (A \smallsetminus B) \cup (B \smallsetminus A)$. ($A \mathbin{\triangle} B$ is known as the ***symmetric difference*** of A and B.)

4. Let A be a subset of some given universal set U. Verify that $\overline{\overline{A}} = A$ and that $A \cup \overline{A} = U$.

5. Verify the second of DeMorgan's Laws for sets, $\overline{A \cap B} = \overline{A} \cup \overline{B}$. For each step in your verification, state why that step is valid.

6. The subset operator, $\subseteq$, is defined in terms of the logical implication operator, $\rightarrow$. However, $\subseteq$ differs from the $\cap$ and $\cup$ operators in that $A \cap B$ and $A \cup B$ are *sets*, while $A \subseteq B$ is a *statement*. So the relationship between $\subseteq$ and $\rightarrow$ isn't quite the same as the relationship between $\cup$ and $\vee$ or between $\cap$ and $\wedge$. Nevertheless, $\subseteq$ and $\rightarrow$ do share some similar properties. This problem shows one example.

- **a)** Show that the following three compound propositions are logically equivalent: $p \rightarrow q$, $(p \wedge q) \leftrightarrow p$, and $(p \vee q) \leftrightarrow q$.
- **b)** Show that for any sets A and B, the following three statements are equivalent: $A \subseteq B$, $A \cap B = A$, and $A \cup B = B$.

7. DeMorgan's Laws apply to subsets of some given universal set U. Show that for a subset X of U, $\overline{X} = U \smallsetminus X$. It follows that DeMorgan's Laws can be written as $U \smallsetminus (A \cup B) = (U \smallsetminus A) \cap (U \smallsetminus B)$ and $U \smallsetminus (A \cap B) = (U \smallsetminus A) \cup (U \smallsetminus B)$. Show that these laws hold whether or not A and B are subsets of U. That is, show that for any sets A, B, and C, $C \smallsetminus (A \cup B) = (C \smallsetminus A) \cap (C \smallsetminus B)$ and $C \smallsetminus (A \cap B) = (C \smallsetminus A) \cup (C \smallsetminus B)$.

8. Show that $A \cup (A \cap B) = A$ for any sets A and B.

9. Let X and Y be sets. Simplify each of the following expressions. Justify each step in the simplification with one of the rules of set theory.

- **a)** $X \cup (Y \cup X)$
- **b)** $(X \cap Y) \cap \overline{X}$
- **c)** $(X \cup Y) \cap \overline{Y}$
- **d)** $(X \cup Y) \cup (X \cap Y)$

10. Let A, B, and C be sets. Simplify each of the following expressions. In your answer, the complement operator should only be applied to the individual sets A, B, and C.

- **a)** $\overline{A \cup B \cup C}$
- **b)** $\overline{A \cup B \cap C}$
- **c)** $\overline{\overline{A \cup B}}$
- **d)** $\overline{B \cap \overline{C}}$
- **e)** $\overline{A \cap \overline{B \cap \overline{C}}}$
- **f)** $A \cap \overline{A \cup B}$