Infinities

- a set A is *countably infinite* if there is a one-to-one correspondence between $\mathbb{N}$ and A
- a set A is *countable* if it is either finite or countably infinite
  - it is possible in principle to make a list of the elements of A
- a set A is *uncountable* otherwise
  - it is impossible to make a list of the elements of A

Infinities

- $\mathbb{Z}$ (integer) is countably infinite

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0  1  -1  2  -2  3  -3  ...
0  1   2  3   4  5   6  ...
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- $\mathbb{Q}$ (rationals) is countably infinite

$$\left(\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \frac{2}{5}, \ldots\right)$$

10. Show that the set $\mathbb{N} \times \mathbb{N}$ is countable.

- use a diagonalization argument

$$\begin{matrix} (0,0) & (0,1) & (0,2) & \cdots \\ (1,0) & (1,1) & (1,2) & \cdots \\ (2,0) & (2,1) & (2,2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

$$(0,0), (0,1), (1,0), (0,2), (1,1), (2,0), (0,3), (1,2), (2,1), (3,0), \ldots$$

a) Suppose that A and B are countably infinite sets. Show that $A \cup B$ is countably infinite.

Infinities

- ℝ (reals) is not countably infinite

0.**9**0398937249879561297927654857945...
0.1**2**349342094059875980239230834549...
0.22**4**0004329843623470932327998957 9...
0.500**0**00000000000000000000000000000...
0.7774**3**449234234876990120909480009...
0.77755**5**555888888889498888980000111...
0.123456**7**888888888888888888000000000...
0.3483544**0**009848712712123940320577...
0.93473244**4**479004983409999990948900...

for any such list, a new number not in the listed can be constructed by picking a number not in bold for each column

- ℝ\ℚ (irrationals) is not countably infinite

> **Theorem 2.9.** *Suppose that X is an uncountable set, and that K is a countable subset of X. Then the set $X \smallsetminus K$ is uncountable.*

Infinities

> **Theorem 2.11.** *Let X be any set. Then there is no one-to-one correspondence between X and $\mathcal{P}(X)$.*

- for finite sets, $|\mathcal{P}(X)| = 2^{|X|} > |X|$
- the "larger" relationship holds for infinite sets too

- can construct an infinite series of increasingly larger infinities with ℝ, $\mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathcal{P}(\mathbb{R}))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))$, …