Suppose that $a_1, a_2, \ldots, a_{10}$ are real numbers, and suppose that $a_1+a_2+\ldots+a_{10} > 100$. Use a proof by contradiction to conclude that at least one of the numbers a_i must be greater than 10.

Answer:

Proof. Assume $a_1 + a_2 + \ldots + a_{10} > 100$. For a proof by contradiction, also assume $\neg\exists i(a_i > 10)$.

$$\neg\exists i(a_i > 10) \equiv \forall i(a_i \leq 10)$$

If all of the $a_i \leq 10$, then $a_1 + \ldots a_{10} \leq 100$. But this contradicts the premise that the sum is greater than 100. Thus the assumption $\neg\exists i(a_i > 10)$ was invalid, and it is the case that $\exists i(a_i > 10)$ i.e. at least one of the a_i must be greater than 10. □

Discussion:

The goal is to show

$$a_1 + a_2 + \ldots + a_{10} > 100 \to \exists i(a_i > 10)$$

We assume the left side ($a_1 + a_2 + \ldots + a_{10} > 100$) and want to show the right side; a proof by contradiction says to then assume the opposite. So we also assume $\neg\exists i(a_i > 10)$.

Simplify:

$$\begin{aligned}\neg\exists i(a_i > 10) &\equiv \forall i\neg(a_i > 10)\\ &\equiv \forall i(a_i \leq 10)\end{aligned}$$

The goal with proof by contradiction is to show a false statement, typically by deriving $\neg q$ for some premise q. Our premise is $a_1 + a_2 + \ldots + a_{10} > 100$, so let's consider what we know about the sum in light of the assumption that $\forall i(a_i \leq 10)$.

If all of the a_i are less than or equal to 10, the sum of 10 of them is going to be less than or equal to 100. But that contradicts the premise that the sum is greater than 100.

Let n be an integer. If n^2 is an even integer, then n is an even integer.

Answer:

Proof. Assume n^2 is an even integer. For a proof by contradiction, also assume that n is an odd integer.

By the definitions of odd numbers and evenly divisible, $n = 2k+1$ for some integer k.

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

$2(2k^2+2k)+1$ is odd, because 2 times anything is even, and so n^2 is odd. But this contradicts the premise that n^2 is even, so we conclude that the assumption that n is odd is invalid and thus if n^2 is an even integer, n is also an even integer. □

Discussion:

The goal is to show

$$n^2 \text{ is an even integer} \rightarrow \text{n is an even integer}$$

We assume the left side (n^2 is an even integer) and want to show the right side; a proof by contradiction says to then assume the opposite. So we also assume that n is an odd integer.

Thus, by the definitions of odd numbers and evenly divisible, $n = 2k + 1$ for some integer k. Since we want to show a contradiction with respect to the premise that n^2 is an even integer, we consider what that means for n^2.

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

$2(2k^2+2k)+1$ is odd, because 2 times anything is even, and so n^2 is odd. This is a contradiction, and we conclude that if n^2 is an even integer, n is also an even integer.