For sets A, B, and C, use the definition of $\subseteq$ and $\cap$ to show that $C \subseteq A \cap B$ if and only iff $(C \subseteq A) \land (C \subseteq B)$.

Answer:

Proof. Show $C \subseteq (A \cap B) \leftrightarrow (C \subseteq A) \land (C \subseteq B)$.

$$C \subseteq (A \cap B) \leftrightarrow (C \subseteq A) \land (C \subseteq B)$$
$$\forall x(x \in C \to x \in (A \cap B)) \leftrightarrow (\forall x(x \in C \to x \in A)) \land (\forall x(x \in C \to x \in B))$$
definition of $\subseteq$
$$\forall x(x \in C \to (x \in A \land x \in B)) \leftrightarrow (\forall x(x \in C \to x \in A)) \land (\forall x(x \in C \to x \in B))$$
definition of $\cap$
$$\forall x(x \notin C \lor (x \in A \land x \in B)) \leftrightarrow (\forall x(x \notin C \lor x \in A)) \land (\forall x(x \notin C \lor x \in B))$$
definition of $\to$
$$\forall x((x \notin C \lor x \in A) \land (x \notin C \lor x \in B)) \leftrightarrow (\forall x(x \notin C \lor x \in A)) \land (\forall x(x \notin C \lor x \in B))$$
distributive law

Observe that this has the form $\forall x(P(x) \land Q(x)) \leftrightarrow (\forall x P(x)) \land (\forall x Q(x))$. Which is true — $p \land q \to p$ and $p \land q \to q$, so $\forall x(P(x) \land Q(x)) \to \forall x P(x)$ and $\forall x(P(x) \land Q(x)) \to \forall x Q(x)$ and thus $\forall x(P(x) \land Q(x)) \to (\forall x P(x)) \land (\forall x Q(x))$. For the other direction, observe that if $P(x)$ is true for every x and $Q(x)$ is true for every x, then $P(x) \land Q(x)$ has to be true for every x.

So, we have that

$$\forall x((x \notin C \lor x \in A) \land (x \notin C \lor x \in B)) \leftrightarrow (\forall x(x \notin C \lor x \in A)) \land (\forall x(x \notin C \lor x \in B))$$

and thus $C \subseteq A \cap B$ iff $(C \subseteq A) \land (C \subseteq B)$. □

Discussion:

We are looking to show logical equivalence between the two sides, so we can try using the definitions of subset and intersection and then simplifying to see if we can get something where the two sides look similar.

Start with $C \subseteq (A \cap B)$:

$$\begin{aligned} C \subseteq (A \cap B) &= \forall x(x \in C \to x \in (A \cap B)) && \text{definition of } \subseteq \\ &= \forall x(x \in C \to (x \in A \land x \in B)) && \text{definition of } \cap \end{aligned}$$

We can drop the $\forall x$ with the statement "Let x be an arbitrary element" (because if the proposition is true for any x, it has to be true for all of them).

$$\begin{aligned} x \in C \to (x \in A \land x \in B) &= x \notin C \lor (x \in A \land x \in B) && \text{definition of } \to \\ &= (x \notin C \lor x \in A) \land (x \notin C \lor x \in B) && \text{distributive law} \end{aligned}$$

This is now as simplified as it can get, so let's turn to the other side:

$$(C \subseteq A) \wedge (C \subseteq B) = (\forall x(x \in C \rightarrow x \in A)) \wedge (\forall x(x \in C \rightarrow x \in B)) \quad \text{definition of } \subseteq$$

Since there are two separate $\forall x$ terms, we don't want to drop the $\forall x$ — x can be an arbitrary element, but it's a separate arbitrary element for each part.

$$(\forall x(x \in C \rightarrow x \in A)) \wedge (\forall x(x \in C \rightarrow x \in B)) = (\forall x(x \notin C \vee x \in A)) \wedge (\forall x(x \notin C \vee x \in B))$$
$$\text{definition of } \rightarrow$$

This is also as simplified as it can get.

Since there's still $\forall x$ in the righthand side, it probably wasn't useful to have dropped it on the left. So we have

$$\forall x((x \notin C \vee x \in A) \wedge (x \notin C \vee x \in B)) \leftrightarrow (\forall x(x \notin C \vee x \in A)) \wedge (\forall x(x \notin C \vee x \in B))$$

Observe that this has the form $\forall x(P(x) \wedge Q(x)) \leftrightarrow (\forall x P(x)) \wedge (\forall x Q(x))$. Which is true — $p \wedge q \rightarrow p$ and $p \wedge q \rightarrow q$, so $\forall x(P(x) \wedge Q(x)) \rightarrow \forall x P(x)$ and $\forall x(P(x) \wedge Q(x)) \rightarrow \forall x Q(x)$ and thus $\forall x(P(x) \wedge Q(x)) \rightarrow (\forall x P(x)) \wedge (\forall x Q(x))$. For the other direction, observe that if $P(x)$ is true for every x and $Q(x)$ is true for every x, then $P(x) \wedge Q(x)$ has to be true for every x.

So, we have that

$$\forall x((x \notin C \vee x \in A) \wedge (x \notin C \vee x \in B)) \leftrightarrow (\forall x(x \notin C \vee x \in A)) \wedge (\forall x(x \notin C \vee x \in B))$$

and thus $C \subseteq A \cap B$ if and only iff $(C \subseteq A) \wedge (C \subseteq B)$.

Use the laws of logic to verify the associative laws for union and intersection. Show that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.

Answer:

[$A \cup (B \cup C) = (A \cup B) \cup C$ only]

Proof. Rewriting the statement $A \cup (B \cup C) = (A \cup B) \cup C$ in predicate logic:

$$\forall x(x \in (A \cup (B \cup C)) \leftrightarrow x \in ((A \cup B) \cup C))$$

Let x be an arbitrary element. The definition of $\cup$ means that this becomes

$$x \in A \vee (x \in B \vee x \in C) \leftrightarrow (x \in A \vee x \in B) \vee x \in C$$

This is just a statement of the associative law for $\vee$, so it is true and thus $A \cup (B \cup C) = (A \cup B) \cup C$ is true. □

Discussion:

Let's start with $A \cup (B \cup C) = (A \cup B) \cup C$. Since we want to use the laws of logic, first turn this into a proposition instead of a statement about sets:

$$\forall x (x \in (A \cup (B \cup C)) \leftrightarrow x \in ((A \cup B) \cup C))$$

Let x be an arbitrary element. Use the definition of $\cup$:

$$x \in (A \cup (B \cup C)) \leftrightarrow x \in ((A \cup B) \cup C)$$
$$x \in A \vee (x \in B \vee x \in C) \leftrightarrow (x \in A \vee x \in B) \vee x \in C$$

This is true — it is just the associative law for $\vee$.

Show that for any sets A and B, $A \subseteq A \cup B$ and $A \cap B \subseteq A$.

Answer:

[$A \subseteq A \cup B$ only]

Proof.

$A \subseteq A \cup B$	
$\forall x (x \in A \rightarrow x \in (A \cup B))$	definition of $\subseteq$
$\forall x (x \in A \rightarrow (x \in A \vee x \in B))$	definition of $\cup$

Let x be an arbitrary element.

$x \in A \rightarrow (x \in A \vee x \in B)$	
$x \notin A \vee (x \in A \vee x \in B)$	definition of $\rightarrow$
$(x \notin A \vee x \in A) \vee x \in B$	associative law
$\mathbb{T} \vee x \in B$	excluded middle
$\mathbb{T}$	

□

Discussion:

[$A \subseteq A \cup B$ only]

The boolean algebra for sets isn't helpful here, since $\subseteq$ is involved. So, apply the definitions of $\subseteq$ and $\cup$ and simplify:

$A \subseteq A \cup B$	
$\forall x (x \in A \rightarrow x \in (A \cup B))$	definition of $\subseteq$
$\forall x (x \in A \rightarrow (x \in A \vee x \in B))$	definition of $\cup$

Let x be an arbitrary element.

$$
\begin{array}{ll}
x \in A \rightarrow (x \in A \vee x \in B) & \\
x \notin A \vee (x \in A \vee x \in B) & \text{definition of } \rightarrow \\
(x \notin A \vee x \in A) \vee x \in B & \text{associative law} \\
\mathbb{T} \vee x \in B & \text{excluded middle} \\
\mathbb{T} &
\end{array}
$$