Graph Traversal

Building blocks and observations –

- Graph ADT provides operations for getting edges incident on a vertex, and end vertices of an edge
  - from a vertex you can find edges, and from an edge you can find the vertex at the other end
- there may be more than one vertex adjacent to another, so you can't just trace through the graph using a single finger to point at where you are – need a container to hold *discovered* vertices

Using a ~~queue~~ stack for the container leads to ~~breadth-first~~ depth-first search.

- however, DFS is typically implemented recursively rather than using a separate stack

Depth-First Search

```
dfs(G,s)
  for each vertex u in V-{s} do
    state[u] = "undiscovered"
    prev[u] = null
  state[s] = "discovered"
  prev[s] = null
  dfshelper(G,s)

dfshelper(G,u)
  process vertex u (early)
  for each edge (u,v) in G.incidentEdges(u) do
    if state[v] = "undiscovered" then
      process edge (u,v)
      state[v] = "discovered"
      prev[v] = u
      dfshelper(G,v)
  state[u] = "processed"
  process vertex u (late)
```

G is the graph, *s* is the starting vertex

this is a generalized form of the algorithm which allows for both early (before visiting incident edges) and late (after visiting incident edges) operations

a vertex is discovered when an incident (incoming) edge has been visited – have found a path from *s* to it

a vertex is processed when all of its incident (outgoing) edges have been visited – have found everything reachable from it

DFS

```
dfs(G,s)
  for each vertex u in V-{s} do
    state[u] = "undiscovered"
    prev[u] = null
  state[s] = "discovered"
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  dfshelper(G,s)

dfshelper(G,u)
  process vertex u (early)
  for each edge (u,v) in G.incidentEdges(u) do
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  state[u] = "processed"
  process vertex u (late)
```

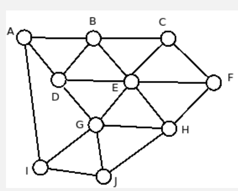

incidentEdges(u) determines what order the edges are visited in

the recursion keeps track of where the algorithm is in the sequence – execution continues when the call returns

Running Time of DFS

total O(n+m) for adjacency list, O(n^2) for adjacency matrix

```
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  for each vertex u in V-{s} do
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```

O(n) with O(n) traversal of vertices and O(1) labeling

incident edges is O(deg(u)) for adjacency list, O(n) for adjacency matrix

total is O(m) for adjacency list (each edge is visited twice, once from each end) and O(n^2) for adjacency matrix (get incident edges once per vertex)

BFS/DFS Search Trees

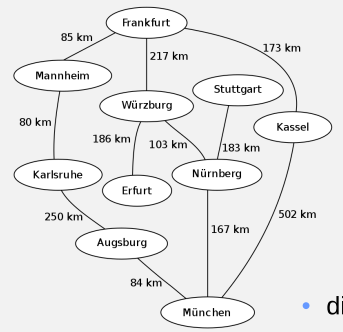
Classify each graph edge (u,v) as it is visited during traversal –

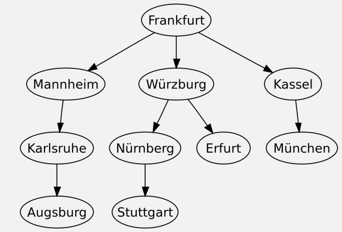
- *discovery* or *tree edges* – v is not already discovered
- *back edges* – v is an ancestor (other than the parent) of u
- *forward edges* – v is a descendant of u
- *cross edges* – v is not an ancestor or a descendant of u

Properties (undirected graphs) –

- discovery (tree) edges form a tree
- non-tree edges in BFS tree are cross edges connecting to the same level or one level higher in another branch
- non-tree edges in DFS tree are back edges

BFS/DFS Search Trees

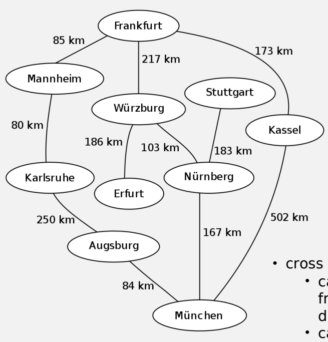


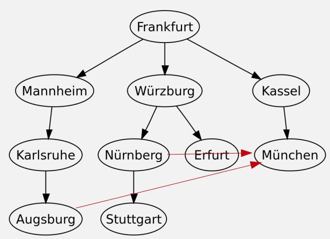


- discovery edges form a tree
  - a newly-discovered vertex is not already part of the tree so it can't be involved in a cycle

BFS/DFS Search Trees

- non-tree edges in BFS tree are cross edges connecting to the same level or one lower in another branch

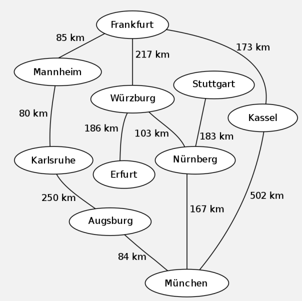


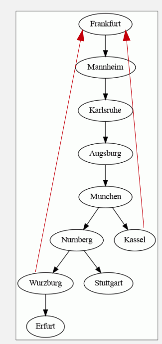


- cross edges can only connect to a vertex in the queue
  - can't connect to a vertex which has been removed from the queue or else it would have been a discovery edge from the other vertex
  - can't connect to a vertex not yet put in the queue or else would be a discovery edge from this vertex
- vertices in the queue at the same time can only be from adjacent levels

BFS/DFS Search Trees

- non-tree edges in DFS tree are back edges





if (u,v) is a non-tree edge, v must be discovered but not processed...

– if v has been processed, all of its incident edges have been visited and thus (v,u) = (u,v) is a discovery edge

...meaning that v is an ancestor of u → (u,v) is a back edge

– if v is the parent of u, (v,u) = (u,v) is a discovery edge

Applications of DFS – Undirected Graphs

- **reachability**
  – every vertex reachable from *s* will be discovered/processed during DFS

intuition – we follow every edge leaving each discovered vertex, and every vertex put in the stack is eventually removed and marked as processed

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Applications of DFS – Undirected Graphs

- **finding cycles**
  – back edge (u,v) forms a cycle consisting of the tree edges from v to u plus back edge (u,v)
  – a graph is a tree if and only if there are no back edges