Dijkstra's Algorithm

```
algorithm dijkstra(G,s):
  for all v in V do
    dist[v] ← ∞
  dist[s] ← 0
  PQ ← makeQueue(V)
  while PQ is not empty do
    v ← PQ.removeMin()
    for each incident edge e=(v,u)
      if dist[u] > dist[v]+w(v,u) then
        dist[u] = dist[v]+w(v,u)
        PQ.decreaseKey(u)
```

Running time.

- initialize dist[v] labels
  - O(n) with O(1) record-keeping and O(1) per element iteration (possible with any graph implementation)
- make PQ with n elements
  - O(makePQ)
- n is-empty checks + n vertices removed from PQ
  - O(n × 1) + O(n × removeMin) [O(1) isEmpty() possible with any PQ implementation]
- 2m (undirected) or m (directed) edges visited over all vertices [2m for undirected because an edge is visited from each of its endpoints]
  - O(m) assuming O(1) per element iteration (possible with adjacency list)
- up to O(1) + O(decreaseKey) per edge (if label is updated)
  - assuming O(1) record-keeping

→ O(makePQ + n × removeMin + m × decreaseKey) total

Dijkstra's Algorithm

→ O(makePQ + n × removeMin + m × decreaseKey) total

operation	heap
makeQueue	O(n log n) – repeated insert O(n) – heapify
removeMin	O(log n)
decreaseKey	O(log n) (**)
total running time for Dijkstra's algorithm	O(n + n log n + m log n) = O((n+m) log n) = O(m log n) for connected graphs

This is –

- O(n log n) for sparse graphs [m = O(n)]
- O(n^2 log n) for dense graphs [m = O(n^2)]

(**) assuming O(1) to locate element within PQ (which can be done with a *locator*), otherwise O(n) to search PQ to find entry

Dijkstra's Algorithm

→ O(makePQ + n × removeMin + m × decreaseKey) total

operation	array - unsorted	heap
makeQueue	O(n) – take elements in whatever order	O(n log n) – repeated insert O(n) – heapify
removeMin	O(n) – search, then swap with last	O(log n)
decreaseKey	O(1) – change key (**)	O(log n) (**)
total running time for Dijkstra's algorithm	O(n + n^2 + m) = O(n^2)	O(n + n log n + m log n) = O((n+m) log n) = O(m log n) for connected graphs

Observation:

- for sparse graphs [m = O(n)], the heap is more efficient
- for dense graphs [m = O(n^2)], the unsorted array is more efficient

(**) assuming O(1) to locate element within PQ (which can be done with a *locator*), otherwise O(n) to search PQ to find entry

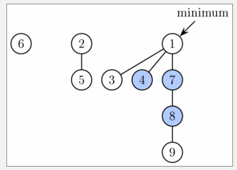
Other Options

(Binary) heaps are not the only choice for implementing priority queues.

- *d*-ary heaps
  - each node has *d* children instead of two, reducing the height of the tree by a factor of log d
  - insert – O(log n / log d)
  - removeMin – O(d log n / log d)
    - have to check all *d* children at each level to determine smallest element
  - *d* should be chosen to be m/n (the average degree of the graph)
  - → total time for Dijkstra's algorithm: O((nd+m) log n / log d) = O(m log n / log (m/n))
    - for sparse graphs [m = O(n)], get O(n log n) – as good as a binary heap
    - for dense graphs [m = O(n^2)], get O(n^2) – as good as an unsorted array
    - in between [m = $n^{1+\delta}$], get O(m) - δ is a constant

Other Options

(Binary) heaps are not the only choice.



- Fibonacci heaps
  - maintain a forest of heaps
  - insert/remove involves splitting and merging heaps in order to keep the degree of each node low and the size of each subtree sufficiently high
  - achieves O(log n) removeMin and O(1) decreaseKey
    - *amortized time* – on average
  - → total time for Dijkstra's algorithm: O(m + n log n)
    - for sparse graphs [m = O(n)], get O(n log n) – as good as a binary heap
    - for dense graphs [m = O(n^2)], get O(n^2) – as good as an unsorted array

Shortest Paths with Negative Edges

Dijkstra's algorithm requires w(u,v) > 0.
But what if there are edges with negative weights?

Approach this like you are dealing with a special case: see where the algorithm you have breaks.

- Dijkstra's algorithm relies on d(s,w) < d(s,v) for all vertices w on the shortest path s → v
- negative or zero weights mean that d(s,w) ≥ d(s,v) is possible
  - thus v may be removed from the PQ before w, and dist[v] is not correct at the time v is marked 'processed' because w has not yet been processed

Bellman-Ford

Idea.

- abandon traversal – instead repeatedly update every edge in the graph

```
algorithm bellmanford(G,s):
  for all v in V do
    dist[v] ← ∞

  dist[s] ← 0

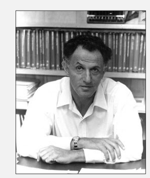
  repeat |V|-1 times
    updated ← false
    for all edges e=(v,u) in E do
      if dist[u] > dist[v]+w(v,u) then
        dist[u] = dist[v]+w(v,u)
        updated ← true
    if !updated then
      break
```

(the longest path from s to any reachable vertex has length n-1, so that many repetitions will serve to propagate dist[s] to every vertex)

(can stop repetitions early if there are no more updates to propagate)

Bellman-Ford(-Moore) algorithm

- L. R. Ford, 1927-2017
- American mathematician also known for
  - network flow problems
  - Ford-Fulkerson algorithm (max flow)
  - Ford-Johnson algorithm (sorting)



- Richard Bellman, 1920-1984
- American applied mathematician also known for
  - introducing dynamic programming (1953)
  - Bellman equation, Hamilton-Jacobi-Bellman equation (optimal control theory)
  - *curse of dimensionality* – exponential increase in volume when adding dimensions
- Edward Moore, 1925-2003
- American professor of mathematics and computer science also known for
  - Moore finite state machine
  - an early pioneer of artificial life
  - Moore graphs
  - Shortest Path Faster Algorithm (variation of Bellman-Ford)

https://en.wikipedia.org/wiki/Richard_E._Bellman

Bellman-Ford

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  repeat |V|-1 times
    updated ← false
    for all edges e=(v,u) in E do
      if dist[u] > dist[v]+w(v,u) then
        dist[u] = dist[v]+w(v,u)
        updated ← true
    if !updated then
      break
```

Running time.

- initialize dist[v] for all vertices
  - O(n) assuming O(1) record-keeping and O(1) per element iteration (possible with any graph implementation)
- for all edges in E, repeated |V|-1 times
  - O(m) ✕ O(n) = O(mn) assuming O(1) record-keeping and O(1) per element iteration (possible with any graph implementation)

→ O(nm) total

Negative Weight Cycles

A possibility with negative weight edges is that there is a negative weight cycle.

- important to detect, because "shortest path" isn't meaningful in that case

Observation: a negative-weight cycle means that there is always at least one edge being updated in a given round of Bellman-Ford.

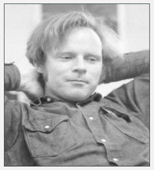
Solution: repeat *n* times. (one extra repetition)

- there is a negative-weight cycle if the loop terminates because the *n*th iteration has been completed instead of because the no-more-updates break has been reached

Other Related Algorithms

- all pairs shortest path – find shortest path from every vertex to every other vertex
  - for each vertex v in V, run Dijkstra's algorithm with v as the starting vertex
    → $O(n^2 \log n)$ to $O(n^3)$ depending on the density of the graph and the PQ implementation
  - Floyd-Warshall algorithm
    → $O(n^3)$
    - lower constant factors than the $O(n^3)$ version of Dijkstra's
    - simpler to implement than Dijkstra's, though finding the shortest path and not just the distance takes more effort

Floyd-Warshall



- Robert Floyd, 1936-2001
- computer scientist also known for
  - Floyd's cycle-finding algorithm
  - Floyd-Steinberg dithering
  - contributions to Hoare Logic
- received the Turing Award in 1978 for contributions relating to the creation of efficient and reliable software

- Stephen Warshall, 1935-2006
- American computer scientist also known for
  - work in operating systems, compilers, language design, and operations research

https://en.wikipedia.org/wiki/Robert_W._Floyd
https://en.wikipedia.org/wiki/Stephen_Warshall

Other Algorithms

- transitive closure – for all pairs of vertices (u,v), determine whether v is reachable from u
  - for each vertex u in V, run bfs(u) to find the reachable vertices
    → $O(n^2 + nm)$
  - run Floyd-Warshall algorithm – v is reachable from u if the length of the shortest path u→v is not ∞
    → $O(n^3)$

Takeaways

- know the algorithm (can trace it), when it is applicable, its running time
  - Dijkstra's algorithm – heap, fancier implementations
  - Bellman-Ford – supports negative weight edges, detects negative weight cycles
- know what it is, suitable algorithms/approaches for solving and their running times
  - all pairs shortest path – repeated Dijkstra, Floyd-Warshall
  - transitive closure – repeated bfs, Floyd-Warshall