## *The Derivative of Other Exponential Functions: $y = b^x$*

Note: The development of the derivative of $y = b^x$ here is different than we did in class using logarithmic differentiation. The result is the same. Logarithmic differentiation is covered on the next page.

When we first determined the derivative of $e^x$ we were looking at general exponential functions of the form $y = f(x) = b^x$. Remember that we picked out $e$ to be the number so that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$ which then made computing the derivative of $e^x$ very easy. But what about all the other exponential functions of the form $y = b^x$? Is there a way to determine their derivatives? Indeed, there is! And it turns out to be pretty easy to do.

The key is to notice that since $e^x$ and $\ln x$ are inverse functions, one undoes the other, so if we apply both of them in succession, first the natural log then the exponential, we end up with the original input. So

$$b^x = e^{\ln b^x} = e^{x \ln b}$$

where we used a log property at the last step to bring the power out front. So $b^x$ is just an exponential function of the form $e^{kx}$ where $k = \ln b$ is a constant [1] So we can find the derivative:

$$\frac{d}{dx}\left(b^x\right) = \frac{d}{dx}\left(e^{x \ln b}\right) = \ln b e^{x \ln b} = (\ln b) b^x = b^x \ln b. \qquad (21.1)$$

[1] This is the great advantage of working with logs: Logs turn products into sums and powers into products. These simplify many calculations.

That was easy. So we have

**THEOREM 21.2** (Derivative of $b^x$). If $b > 0$, then for all $x$,

$$\boxed{\frac{d}{dx}\left(b^x\right) = b^x \ln b.}$$

**EXAMPLE 21.4.** Find the derivatives of the following functions.

$(a)$ $y = 2^x$ $\quad (b)$ $z = 15(3^{t/10})$ $\quad (c)$ $y = 5^{t^3 \sin t}$ $\quad (d)$ $y = 4^x \tan(4x)$

**SOLUTION.** Using Theorem 21.2,

$(a)$ $\frac{d}{dx}\left(2^x\right) = 2^x \ln 2.$

$(b)$ This time we use the chain rule:

$$\frac{d}{dt}\left(15(3^{t/10})\right) = 15(3^{t/10}) \ln 3 \cdot \overbrace{\frac{1}{10}}^{\frac{d}{dt}\left(\frac{t}{10}\right)} = \frac{3}{2}(3^{t/10}) \ln 3.$$

$(c)$ Use the chain rule in combination with Theorem 21.2,

$$\frac{d}{dt}\left(5^{\overbrace{(t^3 \sin t)}^{u}}\right) = 5^{t^3 \sin t} \ln 5 \cdot \overbrace{(3t^2 \sin t - t^3 \cos t)}^{\frac{d}{dt}\left(t^3 \sin t\right)}.$$

$(d)$ Use the product rule:

$$\frac{d}{dx}\left(4^x \tan(4x)\right) = 4^x \ln 4 \cdot \tan(4x) + 4^x \sec^2(4x) \cdot 4 = 4^x(\ln 4 \tan(4x) + 4 \sec^2(4x)).$$

## *Logarithmic Differentiation*

There are still types of functions that we have not tried to differentiate yet. Sometimes we can make use of our existing techniques and clever algebra to find derivatives of very complicated functions. **Logarithmic differentiation** refers to the process of first taking the natural log of a function $y = f(x)$, then solving for the derivative $\frac{dy}{dx}$. On the surface of it, it would seem that logs would only make a complicated function *more* complicated. But remember that logs turn powers into products and products into sums. That's the key.

Let's look at the Extra Credit problem from Exam II to illustrate the idea.

**EXAMPLE 21.5.** Use the chain rule and implicit differentiation along with logs to find the derivative of $y = f(x) = x^x$.

**SOLUTION.** We begin by taking the natural log of both sides and simplifying using log properties.

$$\ln y = \ln x^x \overset{\text{Powers}}{=} x \ln x.$$

Remember we want to find $\frac{dy}{dx}$, so take the derivative of both sides (implicitly on the left).

$$\begin{aligned} \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln x) \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln(x) + 1 \\ \frac{dy}{dx} &\overset{\text{Solve}}{=} y[\ln(x) + 1] \\ \frac{dy}{dx} &\overset{\text{Substitute}}{=} x^x[\ln(x) + 1] \end{aligned}$$

In other words, we have shown that $\frac{d}{dx}(x^x) = x^x[\ln(x) + 1]$. Neat! Easy!

Here are a couple more.

**EXAMPLE 21.6.** Find the derivative of $y = (1 + x^2)^{\tan x}$.

**SOLUTION.** Take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(1 + x^2)^{\tan x} \overset{\text{Powers}}{=} \tan x \ln(1 + x^2).$$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\tan x \ln(1 + x^2)\right) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \sec^2 x \ln(1 + x^2) + \tan x \cdot \frac{2x}{1 + x^2} \\ \frac{dy}{dx} &\overset{\text{Solve}}{=} y\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right] \\ \frac{dy}{dx} &\overset{\text{Substitute}}{=} \ln(1 + x^2)^{\tan x}\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right] \end{aligned}$$

So $\frac{d}{dx}\left(\ln(1 + x^2)^{\tan x}\right) = \ln(1 + x^2)^{\tan x}\left[\sec^2 x \ln(1 + x^2) + \frac{2x \tan x}{1 + x^2}\right]$. Not bad!

**EXAMPLE 21.7.** Find the derivative of $y = (\ln x)^{x^3}$.

**SOLUTION.** Be careful. This function is NOT the same as $\ln(x^{x^3})$ which would equal $x^3 \ln x$. Instead, take the natural log of both sides and simplify using log properties.

$$\ln y = \ln(\ln x)^{x^3} \overset{\text{Powers}}{=} x^3 \ln(\ln x).$$

Do you see the difference when compared to $\ln(x^{x^3})$

Take the derivative of both sides (implicitly on the left) and solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 3x^2\ln(\ln x) + x^3\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$
$$\frac{1}{y}\cdot\frac{dy}{dx} \overset{\text{Solve}}{=} y\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right]$$
$$\frac{dy}{dx} \overset{\text{Substitute}}{=} (\ln x)^{x^3}\left[3x^2\ln(\ln x) + \frac{x^3}{x\ln x}\right]$$

Logs can also be used to simplify products and quotients.

**EXAMPLE 21.8.** Find the derivative of $y = \frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}$.

**SOLUTION.** Use logarithmic differentiation to avoid a complicated quotient rule derivative Take the natural log of both sides and then simplify using log properties.

$$\ln y = \ln\left(\frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\right)$$
$$\overset{\text{Log Prop}}{=} \ln(x^2-1)^5 + \ln(1+x^2)^{1/2} - \ln(x^4+4)$$
$$\overset{\text{Log Prop}}{=} 5\ln(x^2-1) + \frac{1}{2}\ln(1+x^2) - \ln(x^4+4).$$

Take the derivative of both sides and solve for $\frac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}$$
$$\frac{dy}{dx} \overset{\text{Solve}}{=} y\left[\frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}\right]$$
$$\frac{dy}{dx} \overset{\text{Substitute}}{=} \frac{(x^2-1)^5\sqrt{1+x^2}}{x^4+4}\left[\frac{10x}{x^2-1} + \frac{x}{1+x^2} - \frac{4x^3}{x^4+4}\right]$$

That would have been a real mess to do with the quotient rule (which would also require the product rule and the chain rule).

## *Problems*

The following questions will be on the lab tomorrow or are future WebWorK problems. Get a head start.

**1.** Find the derivatives of the following functions. Use logarithmic differentiation where helpful.

(*a*) $y = (\sin x)^x$ (*b*) $y = x^{\sin x}$ (*c*) $(\sin x)^{\sin x}$

(*d*) $(\arcsin x)^{x^2}$ (*e*) $\left(1+\frac{1}{x}\right)^x$

**2.** Find the derivatives of these functions using the derivative formula for a general exponential function that we developed before Exam II. (See Theorem 3.18 on page 194).

(*a*) $5\cdot 6^x$ (*b*) $2^x\cot x$ (*c*) $x^\pi + \pi^x$ (*d*) $x^4\cdot 4^x$

(*e*) For which values of $x$ does $x^4\cdot 4^x$ have a horizontal tangent?

*Answers.*

**1.** (*a*) $\ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln(\sin x) + \frac{x\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^x(\ln(\sin x) + x\cot x)$.

(*b*) $\ln y = \ln x^{\sin x} = \sin x \ln x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln x + (\sin x)\frac{1}{x} \Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\cos x\ln x + \frac{\sin x}{x}\right)$.

(*c*) $\ln y = \ln(\sin x)^{\sin x} = \sin x\ln(\sin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos x\ln(\sin x) + (\sin x)\frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = (\sin x)^{\sin x}\cos x[\ln(\sin x) + 1]$.

(*d*) $\ln y = \ln(\arcsin x)^{x^2} = x^2\ln(\arcsin x) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln(\arcsin x) + x^2\frac{1}{\arcsin x}\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = (\arcsin x)^{x^2}\left(2x\ln(\arcsin x) + \frac{x^2}{(\arcsin x)\sqrt{1-x^2}}\right)$.

(*e*) $\ln y = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) + x\cdot\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\frac{-1}{x^2} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x\left(1+\frac{1}{x}\right)} \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \ln\left(1+\frac{1}{x}\right) - \frac{1}{(x+1)} \Rightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{(x+1)}\right]$.

**2.** (*a*) $\frac{d}{dx}[5\cdot 6^x] = 5\cdot 6^x\ln 6 = 5\ln 6(6^x)$.

(*b*) $\frac{d}{dx}[2^x\cot x] = 2^x\ln 2\cot x - 2^x\csc^2 x = 2^x[\ln 2\cot x - \csc^2 x]$.

(*c*) $\frac{d}{dx}[x^\pi + \pi^x] = \pi x^{\pi-1} + \pi^x\ln\pi$.

(*d*) $\frac{d}{dx}[x^4\cdot 4^x] = 4x^3\cdot 4^x + x^4\cdot 4^x\ln 4 = x^3\cdot 4^x[4 + x\ln 4]$.

(*e*) From the previous part, the slope is 0 when $x^3\cdot 4^x[4+x\ln 4] = 0$. Therefore $x = 0$ or $x = -\frac{4}{\ln 4}$.