The First Derivative Test

Because we know that

- $f'(x) > 0 \Rightarrow f$ is increasing
- $f'(x) < 0 \Rightarrow f$ is decreasing,

we can use this to classify a critical number as either local max or local min or neither as we did in the last example

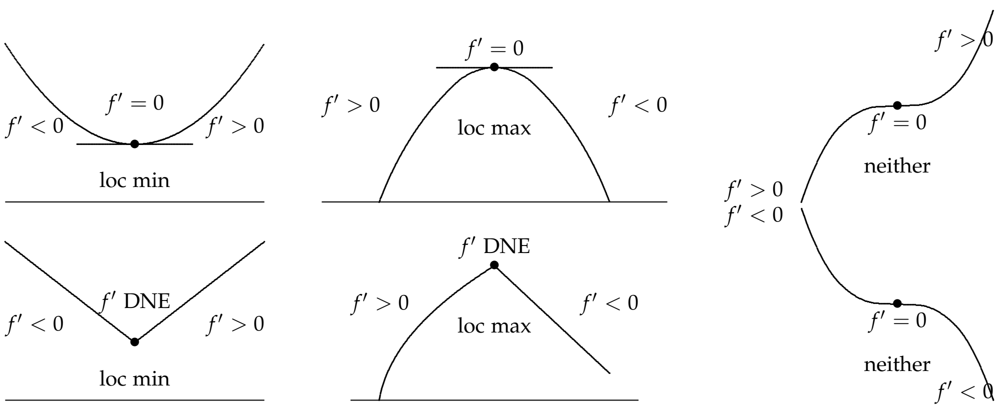


Figure 31.7: When the hypotheses of Rolle's theorem are satisfied, there is a horizontal tangent, i.e., a critical point exists.

Using the intuition from the Increasing/Decreasing Test, we obtain:

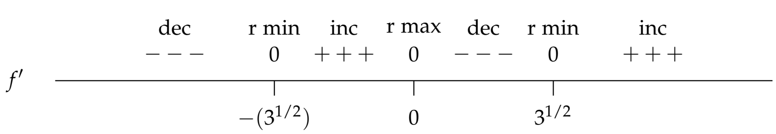
THEOREM 31.8 (The First Derivative Test)**.** Let c be a critical number of a continuous function f.

- If f' changes sign from positive to negative at c, then f has a local max at c.
- If f' changes sign from negative to positive at c, then f has a local min at c.
- If f' does not changes sign at c, then f has neither a local max nor min at c.

YOU TRY IT 31.11. Return to the example at the end of the last section and classify the critical points using the First Derivative Test.

EXAMPLE 31.20. Classify the relative extrema of $f(x) = x^4 - 6x^2 + 1$.

SOLUTION. We determined earlier the sign of $f'(x)$ along a number line. So now all we have to do is fill in the type of critical point.

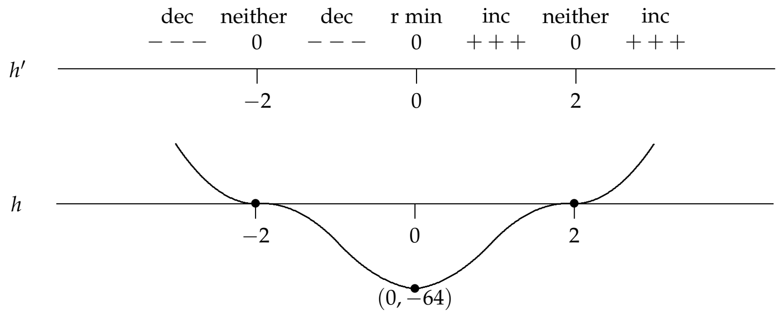


EXAMPLE 31.21. Classify the relative extrema of $h(x) = (x^2 - 4)^3$. Then sketch a quick graph plotting only the critical points.

SOLUTION. Use the First Derivative Test.

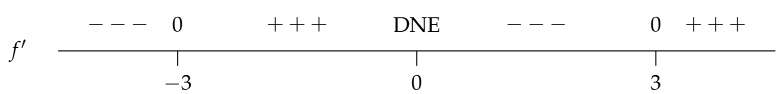
$$h'(x) = 3(x^2 - 4)^2(2x) = 0 \quad \text{at } x = 0, \pm 2.$$

It is easy to determine the sign of the derivative.

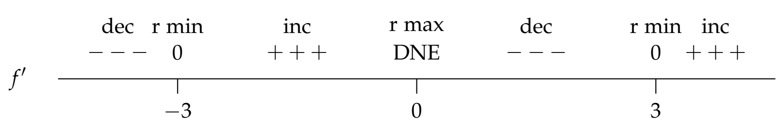


YOU TRY IT 31.12. Classify the relative extrema of $f(x) = \frac{1}{4}(x^2 - 9)^2$. [Answer: The critical values are at $x = 0, \pm 2$. Classify them.]

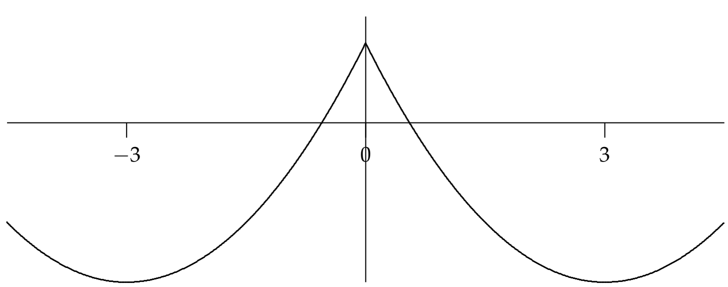
EXAMPLE 31.22. Suppose that a function $f(x)$ is continuous and has the following number line that describes its first derivative. Interpret this information to find where f is increasing, decreasing, and has relative extrema. Then draw a graph of the original function f that satisfies these conditions.



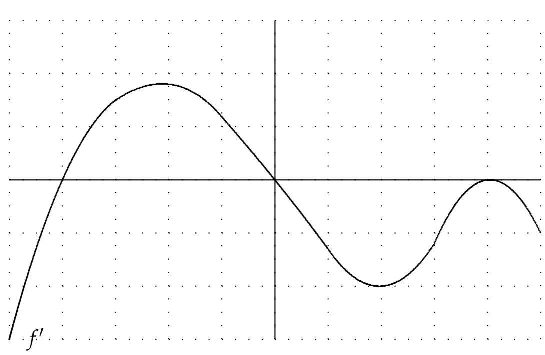
SOLUTION. Use the First Derivative Test to determine what type of extrema we have.



We can use this information to graph one possible solution for f. Note the function is cannot be differentiable at 0 (but should be elsewhere).



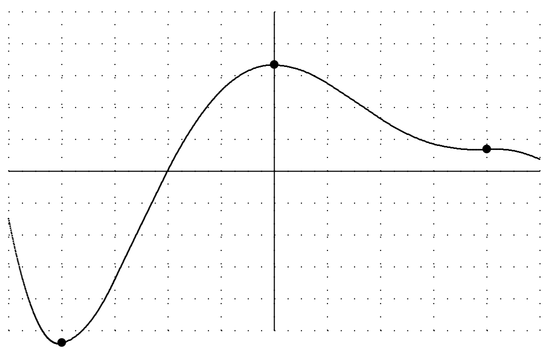
EXAMPLE 31.23. Suppose that the graph of f' is given below. Translate this information into 'number line' form and then attempt to graph the original function f.



SOLUTION. All we need to do is pay attention to the **sign** of the derivative.

dec	r min	inc	r max	dec	none	dec
$--$	0	$+++$	0	$----$	0	$--$
	-4		0		4	

Here's one function that satisfies these conditions. Notice that $x = 4$ is not a relative extreme point.



More Examples

EXAMPLE 31.24. Let $f(x) = xe^{2x}$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical numbers.

$$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}[1 + 2x] = 0 \quad \text{at} \quad x = -1/2.$$

Set up the number line and determine the sign of $f'(x)$ on either side of the critical point. $f'(-1) = -e^{-2} < 0$ and $f'(0) = 1$.

	decreasing	r min	increasing
f'	$---$	0	$+++$
		$-1/2$	

Using interval notation: f is increasing on $(-1/2, \infty)$ and it is decreasing on $(-\infty, -1/2)$. From the First Derivative Test, there is a relative min at $x = -1/2$.

EXAMPLE 31.25. Let $f(x) = x - \sin x$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical numbers.

$$f'(x) = 1 - \cos x = 0 \text{ at } x = 0, \pm 2\pi, \pm 4\pi \ldots.$$

Since $\cos x \leq 1$ the sign of $f'(x)$ between the critical points is always positive. So the $f(x)$ is always increasing and by the First Derivative Test and there are no relative extrema.

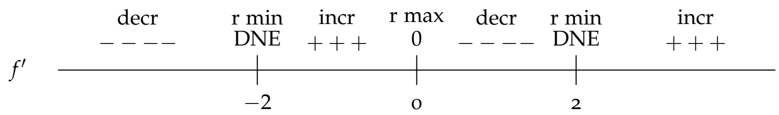
	increase	Not Extr	increase	Not Extr	increase	Not Extr	increase
f'	$+++$	0	$+++$	0	$+++$	0	$+++$
		-2π		0		2π	

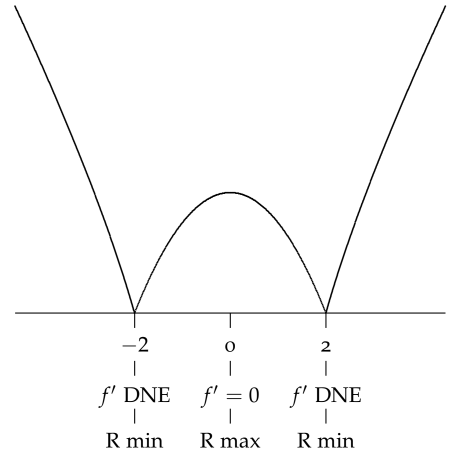
EXAMPLE 31.26. Let $f(x) = (x^2 - 4)^{2/3}$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3} 2x = \frac{4x}{3(x^2 - 4)^{1/3}} = 0 \text{ at } x = 0 \text{ DNE at } x = \pm 2.$$

Determine the sign of $f'(x)$ between and beyond the critical points. $f'(-3) < 0$, $f'(-1) > 0$, $f'(1) < 0$ and $f'(3) > 0$.





Using the First Derivative Test, f has relative mins at $x = \pm 2$ and a relative max at 0. Can you sketch the shape of the graph based on the information?