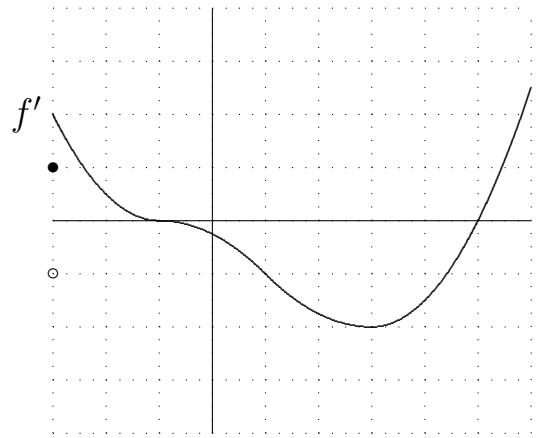
# Math 130: Final Review Questions

**1.** Assume that the graph below is $f'$, the derivative of $f$. Your job is to supply information about the original function $f$.




**a)** Create a number line for $f'$ to determine where $f'$ increases, decreases and has relative extrema.

**b)** By using the slope of $f'$, create a number line for $f''$ and use it to determine concavity and points of inflection?

**c)** If $f$ passes through the point $(-3, 1)$ indicated with a $\bullet$, draw a potential graph of $f$.

**d)** If, instead, $f$ passes through the point $(-3, -1)$ indicated with a $\circ$, draw a potential graph of $f$. What is the relationship between the two graphs you've drawn?

**2.** Limit Review. Several techniques may be used. Show your work. **Remember**: Division by the highest power only applies to limits at infinity. First determine if the limit is indeterminate.

**a)** $\displaystyle\lim_{x\to 4} \frac{x-4}{\sqrt{2x-4}-2}$ **b)** $\displaystyle\lim_{t\to 0^+} \frac{t^2-2}{t^2+2t}$ **c)** $\displaystyle\lim_{t\to 2^+} \frac{t^2-4}{t^2-2t}$ **d)** $\displaystyle\lim_{x\to 0} \frac{\cos x}{x+1}$

**e)** $\displaystyle\lim_{x\to -\infty} \frac{4x-2}{\sqrt{9x^2+1}}$ **f)** $\displaystyle\lim_{x\to +\infty} \frac{2x^3}{3x^{5/3}+10}$ **g)** $\displaystyle\lim_{x\to 0} \frac{2\sin 7x}{\sin 9x}$ **h)** $\displaystyle\lim_{x\to -\infty} \frac{3x^2+7}{9x^2+1}$

**i)** $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$ **j)** $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$ **k)** $\displaystyle\lim_{x\to 2^+} \frac{1+2x}{4-x^2}$ **l)** $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{4x^2+115x}}{3-6x^2}$

**m)** $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{2x}$ **n)** $\displaystyle\lim_{x\to +\infty} \frac{\ln x^2}{x^2}$ **o)** $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 2x}$ **p)** $\displaystyle\lim_{x\to +\infty} \frac{x^2+e^{2x}}{e^{2x}}$

**q)** $\displaystyle\lim_{x\to 2} \frac{\frac{1}{x+3}-\frac{1}{5}}{x-2}$ **r)** $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$ **s)** $\displaystyle\lim_{x\to 2^+} \frac{x^2+1}{x(2-x)}$ **t)** $\displaystyle\lim_{x\to 1^-} \frac{2}{x-1} - \frac{2}{x(x-1)}$

**3.** More limits

**a)** $\displaystyle\lim_{x\to 1} \frac{x-1}{\ln x}$ **b)** $\displaystyle\lim_{x\to 0} \frac{x^2+4x}{\sin 2x}$ **c)** $\displaystyle\lim_{x\to \infty} \frac{6x^2+3x-1}{2x^2+x}$

**d)** $\displaystyle\lim_{x\to \infty} \frac{\ln x}{e^x}$ **e)** $\displaystyle\lim_{x\to 1} \frac{6x+10}{3x+1}$ **f)** $\displaystyle\lim_{x\to 0} \frac{\tan 5x}{\arcsin x}$ **g)** $\displaystyle\lim_{x\to \infty} \left(1+\tfrac{7}{x}\right)^x$

**h)** $\displaystyle\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2}$ **i)** $\displaystyle\lim_{x\to 0} \frac{e^x-1-x}{x^2}$ **j)** $\displaystyle\lim_{x\to 0^+} 2x\ln x$ **k)** $\displaystyle\lim_{x\to \infty} x^2 e^{-x}$

**l)** $\displaystyle\lim_{x\to \infty} \ln(2x^2+9) - \ln(3x^2+27)$ **m)** $\displaystyle\lim_{x\to 0^+} (2x)^x$ **n)** $\displaystyle\lim_{x\to 0^+} (1+x)^{1/x}$ **o)** $\displaystyle\lim_{x\to +\infty} \frac{6x^2-9x}{3x-7x^2}$

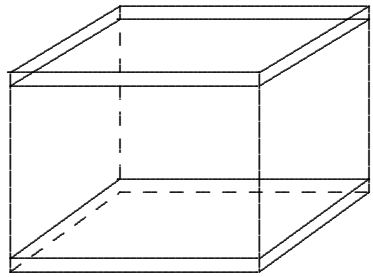
**4.** Does $f(x) = \dfrac{x^2-9}{x^2-4x+3}$ have a vertical asymptote at $x = 3$? Explain.

**5.** Draw a detailed graph of $f(x) = \dfrac{1+x-3x^2}{x^2}$. Include all extrema, inflections, and asymptotes. Show all work including limits to verify the asymptotes.

**6.** **a)** Carefully explain why $f(x) = x^3 + 3x^2 - 9x$ on $[0, 3]$ has both an absolute max and absolute min.

**b)** Find the absolute extreme values of $f(x) = x^3 + 3x^2 - 9x$ on $[0, 3]$.

**7.** A small cardboard **box** with a **square base** is to be constructed to hold a volume of 2,000 in$^3$. It has **double** layers of cardboard on the top and bottom. What dimensions for the box minimize the amount of material used? Show all work. Justify your answer.



**8.** Complete the following definitions:

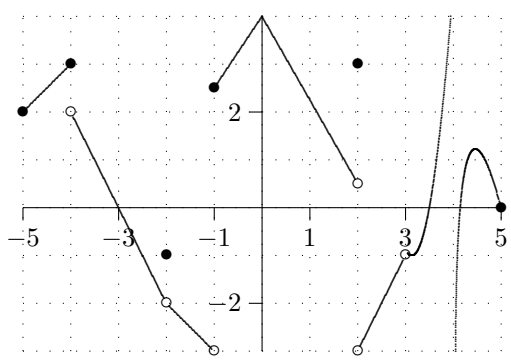
**a)** $y = L$ is a horizontal asymptote of $f$ if:

**b)** $x = a$ is a vertical asymptote of $f$ if:

**c)** $f$ has a removable discontinuity at $x = a$ if:

**d)** $f(x)$ is continuous at $x = a$ if:

**9.** Use the graph of $f$ to evaluate each expression in the table or explain why it does not exist.




| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ |
|---|---|---|---|---|
| $-4$ | | | | |
| $-3$ | | | | |
| $-2$ | | | | |
| $-1$ | | | | |
| 0 | | | | |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | Not Defined | | |
| $-5$ | Not Defined | | | |

**10.** **a)** State the definition of $f$ being *continuous* at $x = a$. Then, use the graph and the table you constructed above to answer these questions.

**b)** Is $f$ continuous at $x = 3.5$? Explain.

**c)** Find a point $x = a$ where $f$ is *not* continuous even though both $f(a)$ and the limit exist.

**d)** Find a point $x = a$ where $f$ is *not* continuous because, though the limit exists, $f(a)$ does not.

**e)** Find a point (not $\pm 5$) where $f$ is *not* continuous because, though $f(a)$ exists, the limit doesn't.

**f)** At which of these points is $f$ continuous: $x = -4, -3, -2, -1, 0, 1, 2$, and 4.

**g)** List 5 other points where $f$ is continuous.

**h)** Find a point where $f$ is continuous but NOT differentiable.

**i)** Find a point where $f$ is differentiable but NOT continuous.

**j)** Find a point where $f$ is differentiable AND continuous.

**k)** At which points is $f$ continuous from the left (but not continuous from both sides).

**11.** Fill in the blanks in the table *consistent with the other information given.* If for some parts more than one value is possible, then use a value of your choice.

| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ | Continuous |
|---|---|---|---|---|---|
| $-3$ | | 1 | | | Yes |
| 1 | | | 2 | | not continuous |
| 3 | | | 0 | 0 | |
| 5 | $-2$ | | | | from the left only |

**12.** **a)** Use the definition of the derivative to find $f'(x)$ for the function $f(x) = x^2 - 4x$. (Use limits.)

**b)** Find the equation of the tangent line to $y = f(x) = x^2 - 4x$ at $x = 3$.

**c)** Carefully use the **limit definition** to find the derivative of $f(x) = \frac{1}{2x}$.

**13.** Define $f(x) = \begin{cases} x^2+1, & \text{if } x \geq 3 \\ 3x+1, & \text{if } x < 3 \end{cases}$. Carefully use limits to answer the following questions.

**a)** Is $f$ continuous at $x = 3$?

**b)** Is $f$ differentiable at $x = 3$?

**14.** Let $s(t) = (t-5)e^t$ be the displacement (meters) or position of an object at time $t$ (seconds). Determine the velocity, acceleration, and the jerk of the object.

**15.** Suppose that $f(x)$ and $g(x)$ are differentiable functions and that $f(2) = -1$, $f'(2) = -2$, $g(3) = 2$, and $g'(3) = -2$. Evaluate each of the following expression.

**a)** $D_x(6x^2 - 3f(x))\Big|_{x=2}$ **b)** $D_x(f(g(x)))\Big|_{x=3}$ **c)** $D_x(f(x^2+x))\Big|_{x=1}$

**16.** Which of the following are polynomials? Rational functions? Neither?

**a)** $\dfrac{2x-4}{x^2-3x+2}$ **b)** $\sqrt{x^3+x} - 2$ **c)** $4x^8 - 9x^4 + 3x$ **d)** $\dfrac{2}{x} - \dfrac{2}{x(x-2)}$

**e)** $2(x^2+3x+1)^{-1}$ **f)** $6x^2 + 64x - 10$ **g)** $\dfrac{1-e^x}{x}$ **h)** $x^{1/2} + x + 1$

**17.** Fill in the *exact* values, if they exist.

| $\theta$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\sin\theta$ | | | | | |
| $\cos\theta$ | | | | | |
| $\tan\theta$ | | | | | |

**18.** Determine the derivatives of each of the following functions.

**a)** $\dfrac{\sqrt[3]{x^5}}{10} + \dfrac{x^4}{10} - \dfrac{2}{3x}$ **b)** $8 + x^3 \sec x$ **c)** $e^{1+\sec(x^2)}$ **d)** $x^3 \arctan(x)$

**e)** $\dfrac{x^2-1}{x^2+1}$ **f)** $\sqrt[4]{5t^3+1}$ **g)** $\cos^3 x - \sin^3 x$ **h)** $(12 + xe^x)^4$

**i)** $e^2 + e^x \tan(e^{3x})$ **j)** $\dfrac{\sin(x)\cos(4x)}{9}$ **k)** $te^{t^2+1}$ **l)** $\arcsin(2x^3)$

**m)** $y = (\cos 9x)^x$ **n)** $y = x^{2\tan x}$ **o)** $x^{\ln x}$

**p)** $(\arcsin x)^{x^2}$ **q)** $\ln(4^{7x^2+11x})$ **r)** $\left(1 - \dfrac{2}{x}\right)^x$

**s)** $5 \cdot 6^x$ **t)** $x^4 \cdot 4^x$ **u)** $3^{x^2+\tan x}$ **v)** $4^{[x^2+1]^3}$

**19. a)** In an animated cartoon, a cylinder is "growing" so that its height is increasing at 2 cm/s and its radius is increasing at 3 cm/s. Determine how the volume of the cylinder ($V = \pi r^2 h$) is changing when the height is 8 cm and the radius is 5 cm. [Ans: $290\pi$ cm$^3$/s]

**b)** How is the surface area ($S = 2\pi rh + 2\pi r^2$) of the cylinder changing? [Ans: $128\pi$ cm$^2$/s]

**20. a)** The bottom of a ladder is sliding away from a wall at 2 ft/s. The ladder is 13 ft tall. How fast is the top of the ladder moving down the wall when the bottom of the ladder is 5 feet away from the wall. [Ans: $-5/6$ ft/s]

**b)** If $\theta$ is the angle between the ladder and the wall, how is $\theta$ changing at the same moment? [Ans: 1/6 rad/s]

**21.** Do a complete graph (critical points, extrema, inflections, increasing, decreasing, concavity, vertical and horizontal asymptotes) of $y = f(x) = \dfrac{4x^2+x-1}{x^2}$. Hint: $f'(x) = \dfrac{-x+2}{x^3}$.

**22.** Graph each of these functions, indicate everything, use number lines, etc.

**a)** $f(x) = x^3 - 3x + 1$ **b)** $g(x) = \dfrac{1}{2}x^4 - 2x$ **c)** $g(x) = \dfrac{1}{2}x^4 - x^2$

**23.** Find the absolute extrema of $f(x) = 1 + x + \cos x$ on $[0, \pi]$. [Ans: 2 and $\pi$]

**24.** Carefully state the Mean Value Theorem and draw a diagram which illustrates it. This is on the exam.

**25. a)** Draw the graph of a continuous function on $[0, 5]$ which has no absolute max or explain why this is impossible.

**b)** Draw the graph of a continuous function on $(0, 5]$ which has no absolute max or explain why this is impossible.

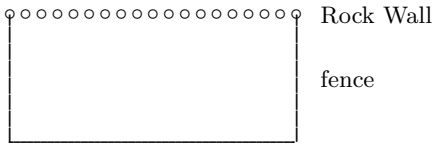
**c)** Draw the graph of a differentiable function on $[0, 5]$ for which $f(0) = -2$ and $f(5) = 1$ and for which $f(x)$ is never equal to 0, or explain why this is impossible.

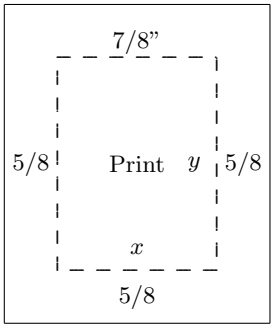
**d)** Make up a graph of a function whose domain is $[-1, 5]$ and which is continuous only from the left at $x = 3$ and neither continuous from the left nor the right at $x = 0$.

**e)** Draw the graph of a function $f$ which satisfies the following conditions or explain why no such function exists: $f'$ $- - - - 0 - - - -$ (at 3) and $f''$ $+ + + + 0 - - - -$ (at 3)

**26.** A calculus student constructs a rectangular garden plot. On two parallel sides, she puts a fence that costs \$12 per meter. On the other pair of parallel sides, she uses a cheaper fence that costs \$8 per meter. Assume that she has \$480 to spend. What is the largest garden area that she can enclose? Be sure to use a theorem to justify your answer.

**27.** A home owner has \$1000 to spend on enclosing her yard. At the front of her house, he wants a rock wall that costs \$18 per meter. On the other three sides he will use fencing that costs \$2 per per meter. What dimensions of the yard *maximize* the area enclosed by the fence? To obtain full credit, completely justify your answer.




**28.** A calculus text has approximately 65 square inches of printed matter on a page with $\frac{5}{8}$ inch margins on each side and the bottom and a $\frac{7}{8}$ inch margin on the top. Find the width $x$ and height $y$ of the printed matter that minimize the total amount of paper used for the page.

**29.** Determine the following antiderivatives:

**a)** $\displaystyle\int 2x^{-1/3} + 3x^{-3/2}\,dx$ **b)** $\displaystyle\int \frac{4}{\sqrt[5]{x^3}} + 3\sqrt{x^5}\,dx$ **c)** $\displaystyle\int \frac{\pi}{\sqrt{1-t^2}}\,dt$

**d)** $\displaystyle\int 4\cos 8\theta - 2\sin(\theta/8)\,d\theta$ **e)** $\displaystyle\int 2\sec t(\tan t - 3\sec t)\,dt$ **f)** $\displaystyle\int \frac{3}{4}e^x - 4x^{-1}\,dx$

**g)** $\displaystyle\int \frac{2x^2+2}{x^{2/3}}\,dx$ **h)** $\displaystyle\int \frac{5}{6x}\,dx$

**30.** **a)** Determine the exact function $f(t)$ such that $f'(t) = e^t + 2t$ and $f(0) = 1$.

**b)** Suppose that $f''(x) = -\frac{2}{x^2}$. If $f'(1) = 4$ and $f(1) = 3$, what is the function $f(x)$?

**c)** Suppose that $f''(x) = 2\sin x$. If $f(0) = 2$ and $f(\pi/2) = \pi$, what is the function $f(x)$?

**31.** My Honda Accord accelerates from 0 to 88 ft/sec (60 mph) in 13 seconds. Assume that the acceleration is constant, $a$.

**a)** Find the particular velocity function of the car.

**b)** Find the position function of the car.

**c)** How far does it travel in this 13 second period?

**32.** A toy rocket is shot upwards from the edge of a canyon (height 180 ft) and hit the canyon floor after 4 seconds. What was the initial velocity of the toy rocket? Remember, $a(t)$ is constant here and $a(t) = -32$ ft/s$^2$.

**33.** On Mars the acceleration due to gravity equals 0.39 that of earth or $-12.5$ ft/sec/sec. Suppose that a Martian calculus student throws his calculus text *upward* at 25 ft/sec off the roof of a building.

**a)** If it takes 6 seconds to hit the ground, how high is the building?

**b)** What was the velocity of the book at the instant it the ground?

**34.** What constant acceleration is required to increase the speed of a car from 30 mph to 50 mph in 5 seconds? Note: 30 mph is 44 ft/s.

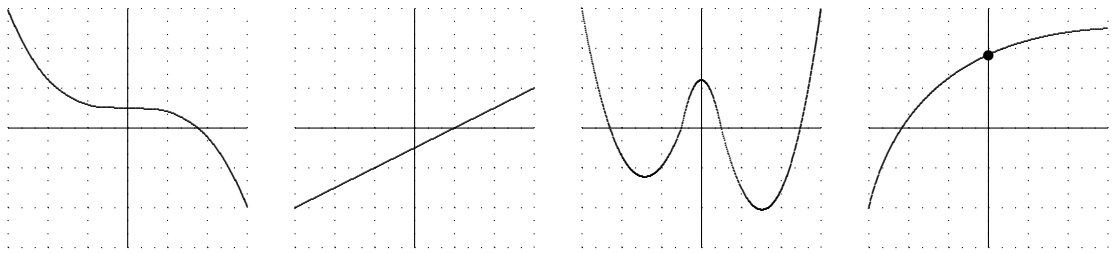
**35.** **a)** A person drops a stone from a bridge. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it? Hint: What's $a(t)$ here (see the rocket problem above). [Ans: 400 ft]

**b)** **Harder Extra Credit**: Did you take into account that sound does not travel instantaneously in your calculation above? Assume that sound travels at 1120 ft/s. What is the height (in feet) of the bridge if the person hears the splash 5 seconds after dropping it? [Ans: 351.68 ft]

**36.** Draw an appropriate right triangle to help evaluate each expression.

**a)** $\sin(\arccos 4x)$ **b)** $\sec(\arctan 3x)$

**37.** Examine the graphs of these functions. Explain which have inverses and which do not. If a function has an inverse, graph it on the same set of axes.



**38.** Find these antiderivatives:

**a)** $\int 9\sin(x/3)\,dx$ **b)** $\int \frac{6}{25+x^2}\,dx$ **c)** $\int \frac{6}{\sqrt{25-x^2}}\,dx$ **d)** $\int (\sqrt[3]{x^2}+\sqrt{x^3})\,dx$

**e)** $\int 3\sec(8x)\tan(8x)\,dx$ **f)** $\int \frac{1+\sqrt{x}}{x}\,dx$ **g)** $\int \frac{e^{2x}-e^{-2x}}{2}\,dx$

**39.** For each relation, find $\frac{dy}{dx}$.

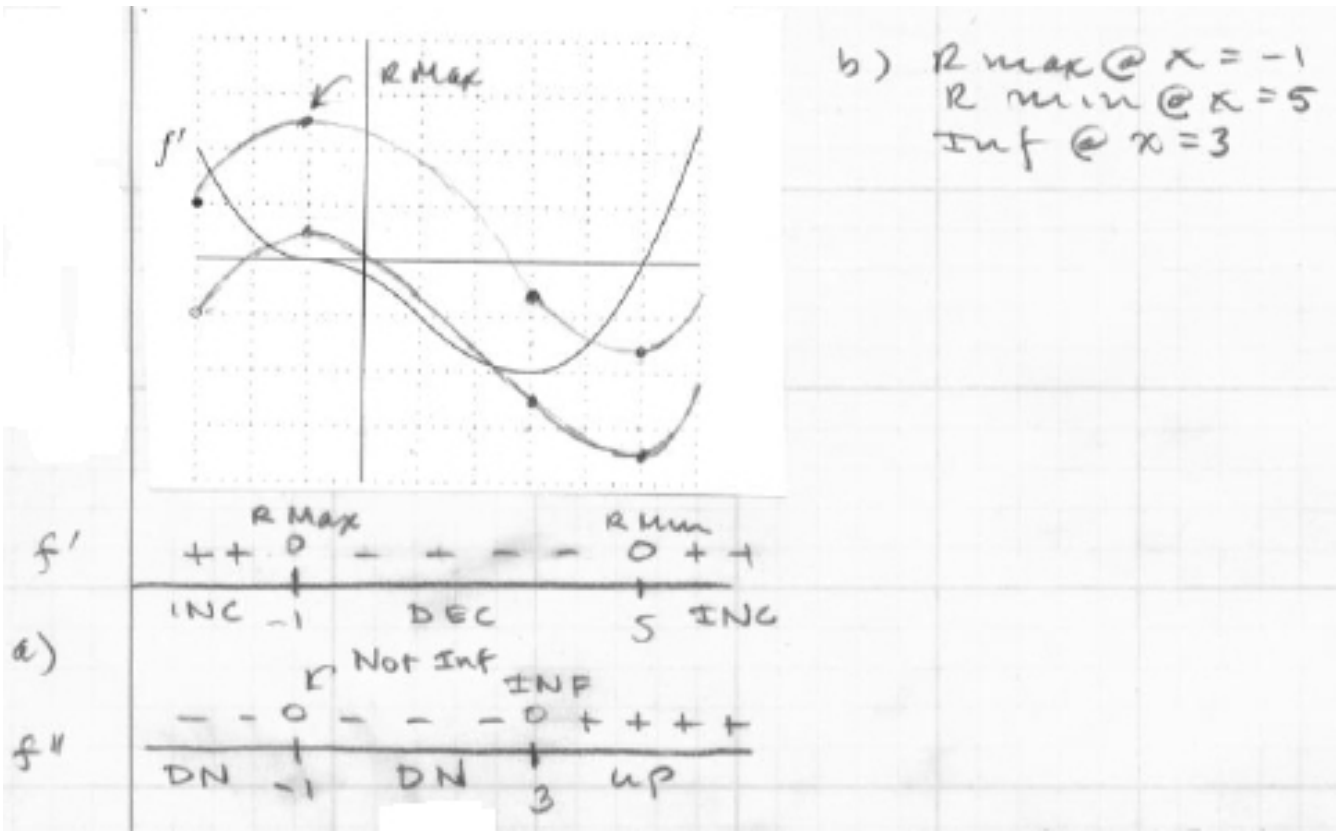
**a)** $y^2 = x^2+1$ **b)** $e^y = \tan x$ **c)** $3y + \ln y = x^2 + x$

**d)** $\sec y - \tan x = 2$ **e)** $x^2 + x^2y^2 + y^3 = 9$ **f)** $xy^2 = 18$ **g)** $\sin(x+y) = 2x$

**h)** Find the tangent line to the curve in (e) at the point $(2,1)$

# Math 130: Answers to Final Review

**1.**




**2.** Some indeterminate forms could also be done by l'Hopital's rule.

**a)** $\frac{0}{0}$; $\overset{\text{l'Ho}}{=} \lim_{x\to 4} \frac{1}{2\cdot\frac{1}{2}(2x-4)^{-1/2}} = \frac{1}{2\cdot\frac{1}{2}(4)^{-1/2}} = 2$

**b)** $\lim_{t\to 0^+} \frac{t^2-2}{t^2+2t}\begin{matrix}\to -2\\ \to 0^+\end{matrix} = -\infty$

**c)** $= \lim_{t\to 2^+} \frac{(t-2)(t+2)}{t(t-2)} = \lim_{t\to 2^+} \frac{t+2}{t} = 2$ or $\overset{\text{l'Ho}}{=} \lim_{\mathrm{t}\to 2^+} \frac{2\mathrm{t}}{2\mathrm{t}-2} = 2$

**d)** $\lim_{x\to 0} \frac{\cos x}{x+1} = \frac{1}{1} = 1$ ctns

**e)** $\frac{-\infty}{\infty}$; $\overset{\text{HP}}{=} \lim_{x\to -\infty} \frac{4x}{\sqrt{9x^2}} = \lim_{x\to -\infty} \frac{4x}{|3x|} = \lim_{x\to -\infty} \frac{4x}{-3x} = -\frac{4}{3}$

**f)** $\frac{\infty}{\infty}$; $\overset{\text{HP}}{=} \lim_{x\to +\infty} \frac{2x^3}{3x^{5/3}} = \lim_{x\to +\infty} \frac{2x^{4/3}}{3}\begin{matrix}\to +\infty\\ \to 3\end{matrix} = +\infty$

**g)** $\frac{0}{0}$; $\overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{14\cos 7x}{9\cos 9x} = \frac{14}{9}$

**h)** $\frac{\infty}{\infty}$; $\overset{\text{HP}}{=} \lim_{x\to -\infty} \frac{3x^2}{9x^2} = \frac{3}{9} = \frac{1}{3}$

**i)** $\frac{0}{0}$; $\overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{2\sec^2 x}{1} = \frac{2}{1} = 2$

**j)** $\frac{0}{0}$; $= \lim_{x\to 3} \frac{\frac{1}{2}(x+1)^{-1/2}}{1} = \lim_{x\to 3} \frac{1}{2\sqrt{x+1}} = \frac{1}{4}$

**k)** $\lim_{x\to 2^+} \frac{1+2x}{4-x^2}\begin{matrix}\to 5\\ \to 0^-\end{matrix} = -\infty$

**l)** $\frac{\infty}{\infty}$; $\overset{\text{HP}}{=} \lim_{x\to -\infty} \frac{\sqrt{4x^2}}{-6x^2} = \lim_{x\to -\infty} \frac{|2x|}{-6x^2} = \lim_{x\to -\infty} \frac{-2x}{-6x^2} = \lim_{x\to -\infty} \frac{-2}{-6x}\begin{matrix}\to -2\\ \to +\infty\end{matrix} = 0$

**m)** $\frac{0}{0}$; $\overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{2x\cos x^2}{2} = \frac{0}{2} = 0$

**n)** $\frac{\infty}{\infty}$; $\overset{\text{l'Ho}}{=} \lim_{x\to +\infty} \frac{\frac{2x}{x^2}}{2x} = \lim_{x\to +\infty} \frac{1}{x^2} = 0$

**o)** $\frac{0}{0}$; $\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{5\cos 5x}{2\cos 2x} = \frac{5}{2}$

**p)** $\frac{\infty}{\infty}$; $\overset{\text{l'Ho}}{=} \lim_{x\to +\infty} \frac{2x+2e^{2x}}{2e^{2x}} \overset{\text{l'Ho}}{=} \lim_{x\to +\infty} \frac{2+4e^{2x}}{4e^{2x}} \overset{\text{l'Ho}}{=} \lim_{x\to +\infty} \frac{8e^{2x}}{8e^{2x}} = 1$

**q)** $\lim_{x\to 2} \frac{\frac{1}{x+3}-\frac{1}{5}}{x-2} = \lim_{x\to 2} \frac{\frac{5-(x+3)}{5(x+3)}}{x-2} = \lim_{x\to 2} \frac{2-x}{5(x+3)(x-2)} = \lim_{x\to 2} \frac{-1}{5(x+3)} = -\frac{1}{25}.$

**r)** $\frac{0}{0}; = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5$

**s)** $\lim_{x\to 2^+} \frac{x^2+1^{\to 5}}{x(2-x)_{\to 0^-}} = -\infty$

**t)** $\frac{0}{0}; = \lim_{x\to 1^-} \frac{2x-2}{x(x-1)} = \lim_{x\to 1^-} \frac{2(x-1)}{x(x-1)} = \lim_{x\to 1^-} \frac{2}{x} = 2$

**3. a)** $\lim_{x\to 1} \frac{x-1}{\ln x} \overset{\text{l'Ho}}{=} \lim_{x\to 1} \frac{1}{1/x} = 1$

**b)** $\lim_{x\to 0} \frac{x^2+4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{2x+4}{2\cos 2x} = \frac{4}{2} = 2$

**c)** $\lim_{x\to\infty} \frac{6x^2+3x-1}{2x^2+x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12x+3}{4x+1} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{12}{4} = 3$

**d)** $\lim_{x\to\infty} \frac{\ln x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{1/x}{e^x} = 0$

**e)** $\lim_{x\to 1} \frac{6x+10}{3x+1} = \frac{16}{4} = 4$

**f)** $\lim_{x\to 0} \frac{\tan 5x}{\arcsin x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{5\sec^2 5x}{\frac{1}{\sqrt{1-x^2}}} = 5$

**g)** Assume $y = \lim_{x\to\infty} \left(1+\frac{7}{x}\right)^x$.

$$\ln y = \ln \lim_{x\to\infty} \left(1+\frac{7}{x}\right)^x = \lim_{x\to\infty} x\ln(1+\frac{7}{x}) = \lim_{x\to\infty} \frac{\ln(1+\frac{7}{x})}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{7}{x}}\cdot\frac{-7}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{7}{x}}\cdot 7 = 7.$$

So $y = e^7 = \lim_{x\to\infty}(1+\frac{7}{x})^x$.

**h)** $\lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-4\sin 4x + 2\sin 2x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{-16\cos 4x + 4\cos 2x}{2} = -\frac{12}{2} = -6.$

**i)** $\lim_{x\to 0} \frac{e^x-1-x}{x^2} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0} \frac{e^x}{2} = 1/2$

**j)** $\lim_{x\to 0^+} 2x\ln x \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2\ln x}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2/x}{-1/x^2} = \lim_{x\to 0^+} -\frac{2x^2}{x} = \lim_{x\to 0^+} -2x = 0$

**k)** $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$

**l)** Use logs: $= \lim_{x\to\infty} \ln \frac{2x^2+9}{3x+27} \overset{\text{HP}}{=} \ln(\lim_{x\to\infty} \frac{2x^2}{3x^2}) \overset{\text{l'Ho}}{=} \ln(\frac{2}{3}) = \ln\frac{2}{3}$

**m)** Assume $y = \lim_{x\to 0^+}(2x)^x$. Then $\ln y = \lim_{x\to 0^+} \ln(2x)^x = \lim_{x\to 0^+} \frac{\ln(2x)}{1/x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{2/2x}{-1/x^2} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0$. So $\ln y = 0$, therefore $y = e^0 = 1 = \lim_{x\to 0^+}(2x)^x$.

**n)** Assume $y = \lim_{x\to 0^+}(1+x)^{1/x}$. As above $\ln y = \lim_{x\to 0^+} \ln(1+x)^{1/x} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$. Therefore $y = e^1 = e = \lim_{x\to 0^+}(1+x)^{1/x}$.

**o)** $\frac{\infty}{-\infty}; \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{6x^2}{-7x^2} = -\frac{6}{7}$

**4.** No, because $\lim_{x\to 3} \frac{x^2-9}{x^2-4x+3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x-1)} = \lim_{x\to 3} \frac{x+3}{x-1} = \frac{6}{2} = 3$. The limit is not infinite.

**5.** $f'(x) = \frac{(1-6x)x^2-(1+x-3x^2)2x}{x^4} = \frac{(1-6x)x-(1+x-3x^2)2}{x^3} = \frac{-2-x}{x^3} = 0$ at $x=-2$, $x=0$ NID.

$f''(x) = \frac{-x^3-(-2-x)3x^2}{x^6} = \frac{-x-(-2-x)(3)}{x^4} = \frac{6+2x}{x^4} = 0$ at $x=-3$, $x=0$ NID.

VA: $\lim_{x\to 0^+} \frac{1+x-3x^2}{x^2} \to \frac{1}{0^+} = +\infty$ and $\lim_{x\to 0^-} \frac{1+x-3x^2}{x^2} \to \frac{1}{0^-} = +\infty$; so VA at $x=0$.

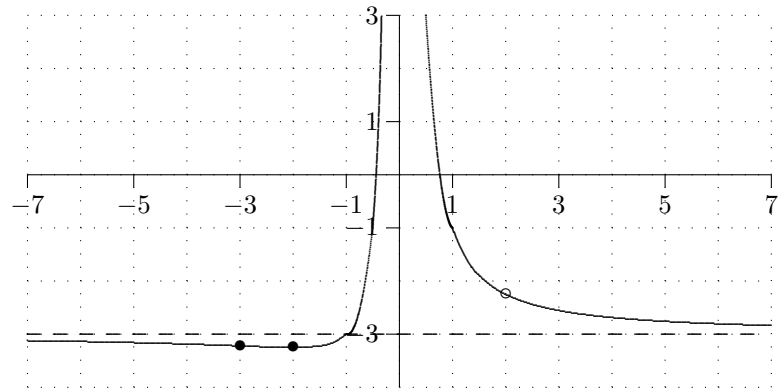
HA: $\lim_{x\to+\infty} \frac{1+x-3x^2}{x^2} = \lim_{x\to+\infty} \frac{\frac{1}{x^2}+\frac{1}{x}-3}{1} = -3$ and $\lim_{x\to-\infty} \frac{1+x-3x^2}{x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}+\frac{1}{x}-3}{1} = -3$.
So HA at $y=-3$.

Evaluate $f$ at key points. $f(-2) = -\frac{13}{4}$, $f(-3) = -\frac{29}{9}$, and another $f(2) = -\frac{9}{4}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $f'$ | $----$ | RMin 0 | $++$ | NID | $--$ | | |
| | Dec | $-2$ | Inc | $0$ | Dec | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $f''$ | $---$ | INF 0 | $++$ | NID | $++$ | | |
| | CD | $-3$ | CU | $0$ | CU | | |

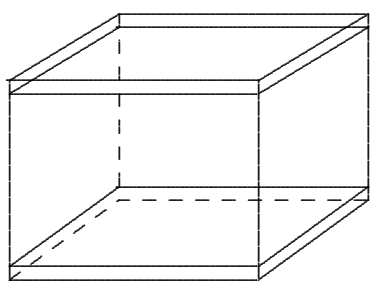
The inflection is almost imperceptible




**6. a)** EVT says a continuous function on a closed interval has both an absolute max and absolute min.

**b)** Use CIT: $f'(x) = 3x^2 + 6x - 9 = x(x^2 + 2x - 3) = (x+3)(x-1) = 0$ at $x = -3$ (not in interval) or $x = 1$. Evaluate $f$ at critical point and endpoints: $f(1) = -5$ (abs min) and $f(0) = 0$ and $f(3) = 27$ (abs max).

**7.** `Minimize` $A = 4x^2 + 4xh$ `(2 tops, 2 bottoms, and 4 sides)`



`Constraint:` $V = x^2 h = 2000$ `in`$^3$

`Eliminate:` $h = \dfrac{2000}{x^2}$ `on` $(0, \infty)$.

`So:` $A = 4x^2 + 4x \cdot \dfrac{2000}{x^2} = 4x^2 + \dfrac{8,000}{x}$ `on` $(0, \infty)$. `Therefore:`

$A' = 8x - \dfrac{8,000}{x^2} = 0 \Rightarrow 8x = \dfrac{8,000}{x^2} \Rightarrow x^3 = 1000 \Rightarrow x = 10.$

`Using the First Derivative Test, there's a RMin at` $x = 10$

| | | | |
|---|---|---|---|
| $f'$ | $----$ | RMin 0 | $++++$ |
| | 0 | 10 | |

`and there's a single critical point. By SCPT, the RMin at` $x = 10$ `is an abs min.` $h = \frac{2000}{(10)^2} = 20$.

**8.** Complete the following definitions:

**a)** $y = L$ is a horizontal asymptote of $f$ if: either

$$\lim_{x \to +\infty} f(x) = L \qquad \text{and/or} \qquad \lim_{x \to -\infty} f(x) = L.$$

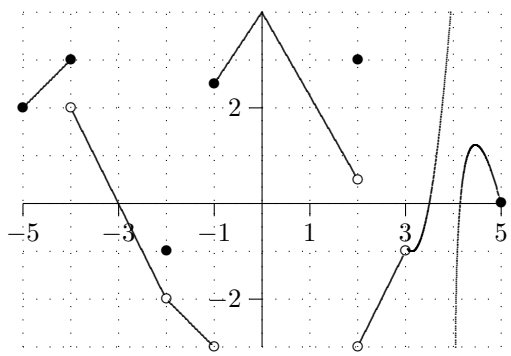
**b)** $x = a$ is a vertical asymptote of $f$ if: either

$$\lim_{x \to a^+} f(x) = +\infty \text{ or } -\infty \qquad \text{and/or} \qquad \lim_{x \to a^-} f(x) = +\infty \text{ or } -\infty.$$

**c)** $f$ has a removable discontinuity at $x = a$ if: $\lim_{x \to a} f(x)$ exists, but does not equal $f(a)$. That is, if $\lim_{x \to a} f(x)$ exists, but $f$ is NOT continuous at $x = a$.

**d)** $f(x)$ is continuous at $x = a$ if: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim_{x \to a} f(x) = f(a)$.

**9.** Use the graph of $f$ to evaluate each expression in the table or explain why it does not exist.




| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ |
|---|---|---|---|---|
| $-4$ | 3 | 2 | DNE | 3 |
| $-3$ | 0 | 0 | 0 | 0 |
| $-2$ | $-2$ | $-2$ | $-2$ | $-1$ |
| $-1$ | $-3$ | 2.5 | DNE | 2.5 |
| 0 | 4 | 4 | 4 | 4 |
| 1 | 2.25 | 2.25 | 2.25 | 2.25 |
| 2 | .5 | $-3$ | DNE | 3 |
| 3 | $-1$ | $-1$ | $-1$ | * |
| 4 | $+\infty$ | $-\infty$ | DNE | * |
| 5 | 0 | Not Defined | * | 0 |
| $-5$ | Not Defined | 2 | * | 2 |

**10. a)** $f$ is *continuous* at $x = a$ if (1) $f(a)$ exists; (2) $\lim_{x\to a} f(x)$ exists; (3) $\lim_{x\to a} f(x) = f(a)$.

**b)** Yes, $\lim_{\to 3.5} f(x) = 1 = f(3.5)$.

**c)** $f$ is *not* continuous even though both $f(a)$ and the limit exist: $x = -2$.

**d)** $f$ is *not* continuous because, though the limit exists, $f(a)$ does not: $x = 3$.

**e)** $f$ is *not* continuous because, though $f(a)$ exists, the limit doesn't: $x = -4, -1, 2$.

**f)** $f$ continuous: $x = -3$, 0, and 1.

**g)** $f$ is continuous: $x \neq -4, -2, -1, 2, 3, 4, (\pm 5)$.

**h)** $f$ is continuous but NOT differentiable: $x = 0$ (corner).

**i)** $f$ is differentiable but NOT continuous. Impossible.

**j)** $f$ is differentiable AND continuous: $x \neq -4, -2, -1, 0, 2, 3, 4, (\pm 5)$.

**k)** $f$ continuous from the left (but not continuous from both sides): $x = -4$.

**11.** Fill in the blanks in the table *consistent with the other information given.* If for some parts more than one value is possible, then use a value of your choice.

| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ | Continuous |
|---|---|---|---|---|---|
| $-3$ | 1 | 1 | 1 | 1 | Yes |
| 1 | 2 | 2 | 2 | not 2 | not continuous |
| 3 | 0 | 0 | 0 | 0 | 0 |
| 5 | $-2$ | not $-2$ | DNE | $-2$ | from the left only |

**12. a)** $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) - (x^2 - 4x)}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - 4x - 4h - x^2 + 4x}{h} = \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h} = \lim_{h\to 0} 2x + h - 4 = 2x - 4.$$

**b)** The general tangent line equation is $y = f(a) + f'(a)(x - a)$. Now $f(3) = -3$ and $f'(3) = 2(3) - 4 = 2$. So the tangent equation is: $y = -3 + 2(x - 3) = 2x - 9$.

**c)** $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} = \lim_{h\to 0} \frac{\frac{2x-2(x+h)}{2(x+h)(2x)}}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{2(x+h)(2x)h} = \lim_{h\to 0} \frac{-1}{(x+h)(2x)} = -\frac{1}{2x^2}$$

**13. a)** Ctns? First $f(3) = 10$. Second: Check one-sided limits

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 + 1 = 10 \text{ and } \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3x + 1 = 10.$$

So $\lim_{x\to 3} f(x) = 10$ and this is the same as $f(3)$ so $f$ is continuous at $x = 3$.

**b)** Differentiable? Use one-sided derivatives:

$$\text{Right}: \lim_{x\to 3^+} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3^+} \frac{x^2 + 1 - 10}{x - 3} = \lim_{x\to 3^+} \frac{(x-3)(x+3)}{x - 3} = \lim_{x\to 3^+} x + 3 = 6.$$

$$\text{Left}: \lim_{x\to 3^-} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to 3^+} \frac{3x + 1 - 10}{x - 3} = \lim_{x\to 3^+} \frac{3x - 9}{x - 3} = \lim_{x\to 3^+} \frac{3(x-3)}{x - 3} = 3.$$

The limits are different so the function is not differentiable at $x = 3$.

**14. a)** $v(t) = s'(t) = e^t + (t-5)e^t = e^t + te^t - 5e^t = te^t - 4e^t = (t-4)e^t$.

**b)** $a(t) = s''(t) = e^t + (t-4)e^t = (t-3)e^t$. **c)** $j(t) = s'''(t) = e^t + (t-3)e^t = (t-2)e^t$.

**15.** Evaluate each of the following expression.

**a)** $D_x(6x^2 - 3f(x))\Big|_{x=2} = 12x - 3f'(x)\Big|_{x=2} = 12(2) - 3f'(2) = 24 - 3(-2) = 30$

**b)** $D_x(f(g(x)))\Big|_{x=3} = f'(g(x))g'(x)\Big|_{x=3} = f'(g(3))g'(3) = f'(2)g'(3) = -2(-2) = 4$

**c)** $D_x(f(x^2 + x))\Big|_{x=1} = f'(x^2 + x)(2x + 1)\Big|_{x=1} = f'(2)(3) = -2(2) = -6$

**16.** Polynomials: c, f. Rational (and not polynomials): a, d, and e.

**17.**

| $\theta$ | $0$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $1/2$ | $\sqrt{2}/2$ | $\sqrt{3}/2$ | $1$ |
| $\cos\theta$ | $1$ | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $0$ |
| $\tan\theta$ | $0$ | $\sqrt{3}/3$ | $1$ | $\sqrt{3}$ | $*$ |

**18.** The derivatives are:

**a)** $\frac{1}{6}x^{2/3} + \frac{2}{5}x^3 + \frac{2}{3}x^{-2}$ **b)** $3x^2 \sec x + x^3 \sec x \tan x$ **c)** $e^{1+\sec(x^2)}(\sec(x^2)\tan(x^2) \cdot 2x)$

**d)** $3x^2 \arctan x + \dfrac{x^3}{1 + x^2}$ **e)** $\dfrac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)} = \dfrac{4x}{(x^2+1)}$ **f)** $\frac{1}{4}(5t^3 + 1)^{-3/4}(15t^2)$

**g)** $3\cos^2 x(-\sin x) - 3\sin^2 x(\cos x) = -3\sin x \cos x(\cos x + \sin x)$ **h)** $4(12 + xe^x)^3(e^x + xe^x)$

**i)** $e^x \tan(e^{3x}) + e^x \sec^2(e^{3x}) \cdot e^{3x} \cdot 3 = e^x[\tan(e^{3x}) + 3e^{3x}\sec^2(e^{3x})$

**j)** $\frac{1}{9}[\cos(x)\cos(4x) - \sin x \sin(4x)(4)]$ **k)** $e^{t^2+1} + te^{t^2+1} \cdot 2t = (1 + 2t^2)e^{t^2+1}$ **l)** $\dfrac{6x^2}{\sqrt{1 - 4x^6}}$

**m)** $\ln y = \ln(\cos 9x)^x = x\ln(\cos 9x) \Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \ln(\cos 9x) - \dfrac{x\sin(9x) \cdot 9}{\cos 9x} \Rightarrow \dfrac{dy}{dx} = (\cos 9x)^x(\ln(\cos 9x) - 9x\tan 9x)$.

**n)** $\ln y = \ln x^{2\tan x} = 2\tan x \ln x \Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = 2\sec^2 x \ln x + (2\tan x)\dfrac{1}{x} \Rightarrow \dfrac{dy}{dx} = x^{2\tan x}\left(2\sec^2 x \ln x + \dfrac{2\tan x}{x}\right)$.

**o)** $\ln y = \ln x^{\ln x} = \ln x \ln x = (\ln x)^2 \Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = 2(\ln x) \cdot \dfrac{1}{x} \Rightarrow \dfrac{dy}{dx} = x^{\ln x} \cdot \dfrac{2\ln x}{x}$.

**p)** $\ln y = \ln(\arcsin x)^{x^2} = x^2 \ln(\arcsin x) \Rightarrow \dfrac{1}{y} \cdot \dfrac{dy}{dx} = 2x\ln(\arcsin x) + x^2 \cdot \dfrac{1}{\arcsin x} \cdot \dfrac{1}{\sqrt{1 - x^2}} \Rightarrow \dfrac{dy}{dx}$

$$= (\arcsin x)^{x^2}\left(2x\ln(\arcsin x) + \frac{x^2}{\arcsin x \cdot \sqrt{1 - x^2}}\right).$$

**q)** $y = \ln(4^{7x^2+11x}) = (7x^2+11x)\ln 4 \Rightarrow y' = (14x+11)\ln 4.$

**r)** $\ln y = \ln\left(1-\dfrac{2}{x}\right)^x = x\ln\left(1-\dfrac{2}{x}\right) \Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln\left(1-\dfrac{2}{x}\right) + x\cdot\dfrac{1}{\left(1-\frac{2}{x}\right)}\cdot\dfrac{2}{x^2} \Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln\left(1-\dfrac{2}{x}\right) + \dfrac{2}{x\left(1-\frac{2}{x}\right)} \Rightarrow \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln\left(1-\dfrac{2}{x}\right) + \dfrac{2}{(x-2)} \Rightarrow \dfrac{dy}{dx} = \left(1-\dfrac{2}{x}\right)^x\left[\ln\left(1-\dfrac{2}{x}\right) + \dfrac{2}{(x-2)}\right].$

**s)** $5\ln 6(6^x)$

**t)** $4x^3\cdot 4^x + x^4\cdot 4^x\ln 4 = x^3\cdot 4^x[4+x\ln 4]$

**u)** $3^{x^2+\tan x}\cdot\ln 3\cdot(2x+\sec^2 x)$

**v)** $6x[x^2+1]^2 4^{[x^2+1]^3}\ln 4$

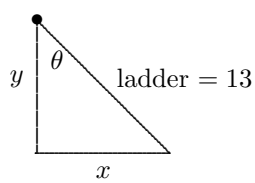
**19. a)** Find $\left.\frac{dV}{dt}\right|_{r=5,\ h=8}$ given $\frac{dr}{dt} = 3$ cm/s and $\frac{dh}{dt} = 2$ cm/s. Relation: $V = \pi r^2 h$. Rate: $\frac{dV}{dt} = 2\pi r\frac{dr}{dt}h + \pi r^2\frac{dh}{dt}$. So

$$\left.\frac{dV}{dt}\right|_{r=5,\ h=8} = 2\pi(5)(3)(8) + \pi(5)^2(2) = 290\pi\ \text{cm}^3/\text{s}.$$

**b)** Find $\left.\frac{dA}{dt}\right|_{r=5,\ h=8}$ given $\frac{dr}{dt} = 3$ cm/s and $\frac{dh}{dt} = 2$ cm/s. Relation: $A = 2\pi rh + 2\pi r^2$. Rate: $\frac{dA}{dt} = 2\pi\frac{dr}{dt}h + 2\pi r\frac{dh}{dt} + 4\pi r\frac{dr}{dt}$. So

$$\left.\frac{dA}{dt}\right|_{r=5,\ h=8} = 2\pi(3)(8) + 2\pi(5)(2) + 4\pi(5)(3) = 128\pi\ \text{cm}^2/\text{s}.$$

**20. a)** `Find` $\left.\frac{dy}{dt}\right|_{x=5}$ `given` $\frac{dx}{dt} = 2$ `ft/s. Relation:` $13^3 = x^2 + y^2$.




`Rate:` $0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow y\frac{dy}{dt} = -x\frac{dx}{dt}$.

`When` $x = 5$, `then` $y = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$.

$$12\left.\frac{dy}{dt}\right|_{x=5} = -5(2) \Rightarrow \left.\frac{dy}{dt}\right|_{x=5} = -\frac{5}{6}\text{ft/s}.$$

**b)** Find $\left.\frac{d\theta}{dt}\right|_{x=5}$ given $\frac{dx}{dt} = 2$ ft/s. Relation: $\sin\theta = \frac{x}{13}$. Now be smart—use what we learned later in the term. Use the inverse trig functions: $\theta = \arcsin\frac{x}{13}$

$$\left.\frac{d\theta}{dt}\right|_{x=5} = \frac{1}{\sqrt{1-(\frac{x}{13})^2}}\cdot\frac{1}{13}\cdot\frac{dx}{dt} = \frac{1}{\sqrt{1-(\frac{5}{13})^2}}\cdot\frac{1}{13}\cdot 2 = \frac{1}{\sqrt{\frac{169-25}{169}}}\cdot\frac{2}{13} = \frac{13}{12}\cdot\frac{2}{13} = \frac{1}{6}\ \text{rad/s}.$$

**21.** $f'(x) = \dfrac{(8x+1)x^2 - (4x^2+x-1)2x}{x^4} = \dfrac{(8x+1)x - (4x^2+x-1)2}{x^3} = \dfrac{2-x}{x^3} = 0$ at $x = 2$, DNE at 0.

$f''(x) = \dfrac{-x^3 - (2-x)3x^2}{x^6} = \dfrac{-x-(2-x)(3)}{x^4} = \dfrac{2x-6}{x^4} = 0$ at $x = 3$, DNE at 0.

VA: $\lim\limits_{x\to 0^+}\dfrac{4x^2+x-1}{x^2} \to \dfrac{-1}{0^+} = -\infty$ and $\lim\limits_{x\to 0^-}\dfrac{4x^2+x-1}{x^2} \to \dfrac{-1}{0^+} = -\infty$; so VA at $x = 0$.

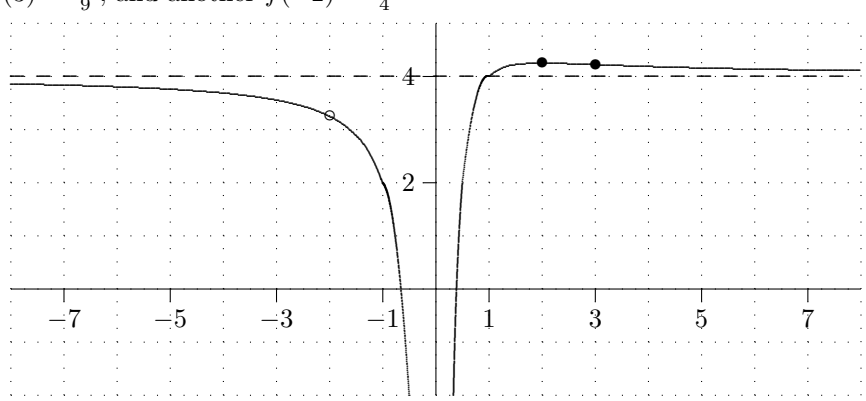
HA: $\lim\limits_{x\to+\infty}\dfrac{4x^2+x-1}{x^2} = \lim\limits_{x\to+\infty}\dfrac{4+\frac{1}{x}-\frac{1}{x^2}}{1} = 4$ and $\lim\limits_{x\to-\infty}\dfrac{4x^2+x-1}{x^2} = \lim\limits_{x\to-\infty}\dfrac{4+\frac{1}{x}-\frac{1}{x^2}}{1} = 4$.
So HA at $y = 4$.

Evaluate $f$ at key points. $f(2) = \frac{17}{4}$, $f(3) = \frac{39}{9}$, and another $f(-2) = \frac{13}{4}$

The inflection is almost imperceptible

| | | | | | |
|---|---|---|---|---|---|
| $f'$ | $--$ | NID | $++$ | RMax 0 | $--$ |
| | Dec | 0 | Inc | 2 | Dec |

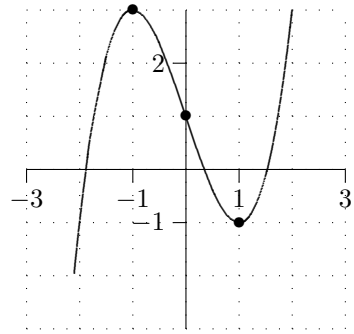
| | | | | | |
|---|---|---|---|---|---|
| $f''$ | $--$ | NID | $--$ | INF 0 | $++$ |
| | CD | 0 | CD | 3 | CU |



**22. a)** Graph $f(x) = x^3 - 3x + 1$. $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x = 0 \Rightarrow x = 0$. No VA or HA. Evaluate $f$ at key points. $f(-1) = 3$, $f(0) = 1$, $f(1) = -1$

| | | RMax | | Rmin | |
|---|---|---|---|---|---|
| $f'$ | $++$ | $0$ | $--$ | $0$ | $++$ |
| | Inc | $-1$ | Dec | $1$ | Inc |

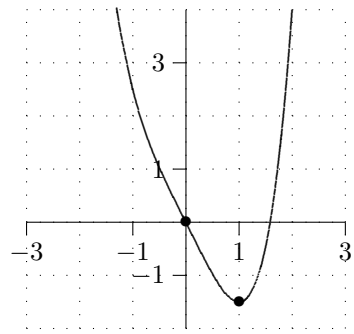
| | | INF | |
|---|---|---|---|
| $f''$ | $--$ | $0$ | $++$ |
| | CD | $0$ | CU |




**b)** Graph $g(x) = \frac{1}{2}x^4 - 2x$. $g'(x) = 2x^3 - 2 = 2(x^3 - 1) = 0 \Rightarrow x = 1$. $g''(x) = 6x^2 = 0 \Rightarrow x = 0$. No VA or HA. Evaluate $g$ at key points. $g(1) = -1.5$ and $g(0) = 0$

| | | Rmin | |
|---|---|---|---|
| $g'$ | $---$ | $0$ | $++$ |
| | Dec | $1$ | Inc |

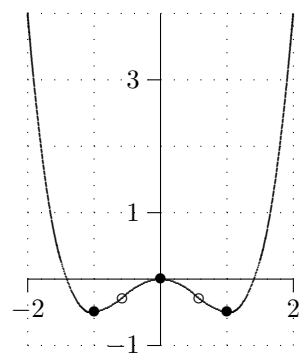
| | | NO Inf | |
|---|---|---|---|
| $g''$ | $+++$ | $0$ | $+++$ |
| | CU | $0$ | CU |




**c)** Graph $g(x) = \frac{1}{2}x^4 - x^2$. $g'(x) = 2x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$. $g''(x) = 6x^2 - 2 = 6(x^2 - \frac{1}{3} = 0 \Rightarrow x = \pm\sqrt{\frac{1}{3}}$. No VA or HA. Evaluate $g$ at key points. $g(0) = 0$, $g(1) = \frac{1}{2}$, $g(-1) = -\frac{1}{2}$, $g\left(\sqrt{\frac{1}{3}}\right) = -\frac{5}{18}$, $g\left(-\sqrt{\frac{1}{3}}\right) = -\frac{5}{18}$,

| | | Rmin | | Rmax | | Rmin | |
|---|---|---|---|---|---|---|---|
| $g'$ | $--$ | $0$ | $++$ | $0$ | $--$ | $0$ | $++$ |
| | Dec | $-1$ | Inc | $0$ | Dec | $1$ | Inc |

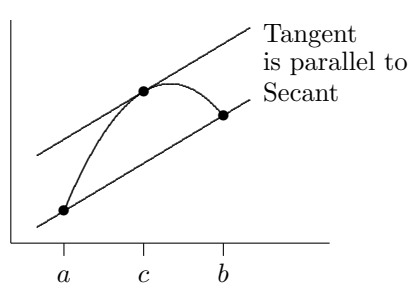
| | | Inf | | Inf | |
|---|---|---|---|---|---|
| $g''$ | $+++$ | $0$ | $---$ | $0$ | $+++$ |
| | CU | $-(\frac{1}{3})^{1/2}$ | CD | $(\frac{1}{3})^{1/2}$ | CU |



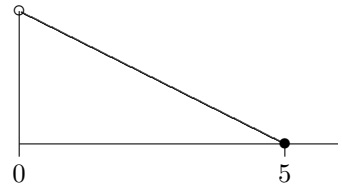

**23.** Use CIT. $f'(x) = 1 - \cos x \Rightarrow \sin x = 1 \Rightarrow x = \pi/2$. Check $f$ at critical points and endpoints: $f(\pi/2) = 1 + \pi/2$. $f(0) = 1$ (abs min). $f(\pi) = \pi$ (abs max).

**24.** Let $f$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval $[a, b]$. Then there is some point $c$ between $a$ and $b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



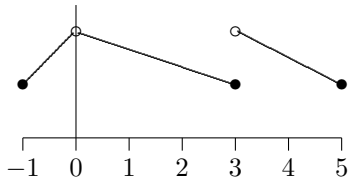
**25.** **a)** Impossible. Differentiable $\Rightarrow$ Continuous. The EVT applies. A continuous function on a closed interval must have an absolute max.




**b)** Since the interval is not closed, the EVT does NOT apply.

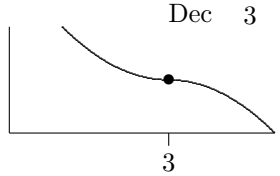
**c)** Impossible. Differentiable $\Rightarrow$ Continuous. Then IVT applies. Since the interval is closed and $f(0) < 0 < f(5)$, then there must be some point $c$ between 0 and 5 so that $f(c) = 0$.

**d)** Possible

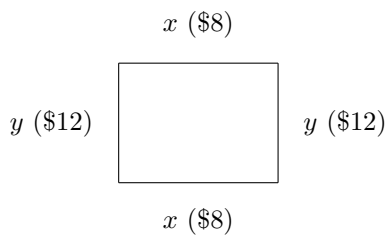



**e)** Possible: $f'$ $- - - - 0 - - - -$ Dec 3 Dec and $f''$ $+ + + + 0 - - - -$ Dec 3 Inc

so




**26.** `Maximize` $A = xy$.




`Constraint:` $480 = 8(2x) + 12(2y) = 16x + 24y$ `dollars`

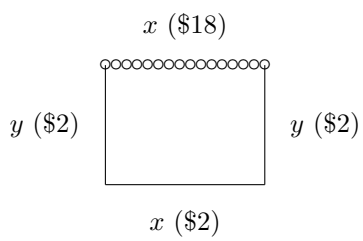
`Eliminate:` $16x = 480 - 24y \Rightarrow x = 30 - 1.5y$ `on` $[0, 20]$.

`So:` $A = xy = (30 - 1.5y)y = 30y - 1.5y^2$ `on` $[0, 20]$. `Therefore:`

$A' = 30 - 3y = 0 \Rightarrow y = 10$.

`Use CIT. Check values at critical and end points.` $A(10) = 150$ `sq. ft. (Abs Max).` $A(0) = 0$ `and` $A(20) = 0$.

**27.** `Maximize` $A = xy$.




`Constraint:` $1000 = 18x + 2x + 2(2y) = 20x + 4y$ `dollars`

`Eliminate:` $4y = 1000 - 20x \Rightarrow y = 250 - 5x$ `on` $[0, 50]$.

`So:` $A = xy = x(250 - 5x) = 250x - 5x^2$ `on` $[0, 50]$. `Therefore:`

$A' = 250 - 10x = 0 \Rightarrow x = 25$.

`Use CIT. Check values at critical and end points.` $A(25) = 3125$ `sq. ft. (Abs Max).` $A(0) = 0$ `and` $A(50) = 0$.

**28.** Minimize paper $TA = (x + \frac{5}{4})(y + \frac{3}{2})$.
Constraint: $PA = xy = 65$
Eliminate: $x = \dfrac{65}{y}$ on $(0, \infty)$.
So: $TA = (\frac{65}{y} + \frac{5}{4})(y + \frac{3}{2}) = 65 + \frac{195}{2y} + \frac{5y}{4} + \frac{15}{18}$ on $(0, \infty)$. Therefore:

$$TA' = -\frac{195}{2y^2} + \frac{5}{4}. \Rightarrow \frac{10}{4}y^2 = 195 \Rightarrow y^2 = 78 \Rightarrow y = \sqrt{78}$$

Using the First Derivative Test, there's a RMin at $y = \sqrt{78}$: $f'$ $- - - -$ RMin $0$ $+ + + +$ 0 $(78)^{1/2}$

Since there's a single critical point, by SCPT the RMin at $y = \sqrt{78}$ is an abs min. $x = \frac{65}{\sqrt{78}}$.

**29.** The antiderivatives are

**a)** $3x^{2/3} - 6x^{-1/2} + c$ **b)** $= \int 4x^{-3/5} + 3x^{5/2}\,dx = 10x^{2/5} + \dfrac{6}{7} \cdot x^{7/2} + c$ **c)** $\pi \arcsin t + c$

**d)** $\dfrac{1}{2}\sin 8\theta + 16\cos\theta + c$ **e)** $= \int 2\sec t \tan t - 6\sec^2 t\,dt = 2\sec t - 6\tan t + c$ **f)** $\dfrac{3}{4}e^x - 4\ln|x| + c$

**g)** $= \int 2x^{4/3} + 2x^{-2/3}\,dx = \dfrac{6}{7} \cdot x^{7/3} + 6x^{1/3} + c$ **h)** $\dfrac{6}{5}\int \dfrac{1}{x}\,dx = \dfrac{6}{5}\ln|x| + c$

**30. a)** $f(t) = \int f'(t)\,dt = \int e^t + 2t\,dt = e^t + t^2 + c$ and $f(0) = 1 + 0 + c = 1$. So $c = 0$ and $f(t) = e^t + t^2$.

**b)** $f'(x) = \int -\frac{2}{x^2}\,dx = \frac{2}{x} + c$. $f'(1) = 2 + c = 4 \Rightarrow c = 2$, so $f'(x) = \frac{2}{x} + 2$. $f(x) = \int \frac{2}{x} + 2\,dx = \ln x + 2x + c$ and $f(1) = 0 + 2 + c = 3 \Rightarrow c = 1$. Thus, $f(x) = \ln x + 2x + 1$.

**c)** $f'(x) = \int 2\sin x\,dx = -2\cos x + c$. So $f(x) = \int -2\cos x + c\,dx = -2\sin x + cx + d$. $f(0) = 0 + 0 + d = 2$, so $d = 2$. Now $f(\pi/2) = 1 + (\pi/2)c + 2 = \pi \Rightarrow (\pi/2)c = -3$, so $c = -\frac{6}{\pi}$. So $f(x) = -2\sin x - \frac{6}{\pi}x + 2$.

**31.** Given: $a(t) = s''(t) = a$ constant. $v(0) = 0$ ft/s. $s(0) = 0$. And $v(13) = 88$ ft/s.

**a)** $v(t) = \int a\,dt = at + c$. But $v(0) = a \cdot 0 + c = 0 \Rightarrow c = 0$. So $v(t) = at$. But $v(13) = 13a = 88 \Rightarrow a = \frac{88}{13}$. So $v(t) = \frac{88}{13}t$.

**b)** $s(t) = \int \frac{88}{13}t\,dt = -\frac{44}{13}t^2 + c$. But $s(0) = 0 + c = 0 \Rightarrow c = 0$. So $s(t) = -\frac{44}{13}t^2$.

**c)** $s(13) = \frac{44}{13}(13)^2 = 572$ ft.

**32. a)** Given: $a(t) = -32$ ft/s$^2$ constant, $s(0) = 180$ ft and $s(4) = 0$ ft. Find $v(0) = s'(0)$ We have $s'(t) = \int -32\,dt = -32t + c$. Can't evaluate $c$ yet. So find $s(t)$ to eventually determine $c$. $s(t) = \int -32t + c\,dt = -16t^2 + ct + d$. But $s(0) = 0 + 0 + d = 180 \Rightarrow d = 180$ so $s(t) = -16t^2 + ct + 180$. Next

$$s(4) = -16(4)^2 + 4c + 180 = 0 \Rightarrow 4c = 256 - 180 = 76 \Rightarrow c = 19 \text{ ft/s}.$$

So now $v(t) = s'(t) = -32t + 19$ so $s'(0) = 19$ ft/s.

**33. a)** Given $a(t) = -12.5$ ft/sec/sec is constant, and $v(0) = 25$ ft/sec and $s(6) = 0$ (hits ground at $t = 6$. How high is the building? Must find $s(0)$ the position at the start. $v(t) = \int -12.5\,dt = -12.5t + c$. $v(0) = 0 + c = 25$, so $v(t) = -12.5t + 25$. $s(t) = \int -12.5t + 25\,dt = -6.25t^2 + 25t + c$. Now set $s(6) = 0 = -6.25(6)^2 + 25(6) + c \to c = 75$ ft. So $s(t) = -6.25t^2 + 25t + 75$ so $s(0) = 75$ ft.

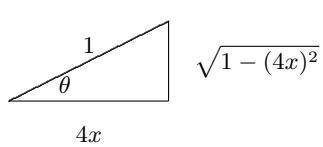
**b)** $v(t) = -12.5t + 25$. So $v(6) = -50$ ft/s.

**34.** Given: $a(t) = s''(t) = a$ constant. $v(0) = s'(0) = 44$ ft/s. $s_0 = 0$. And 50 mph is $\frac{220}{3}$ ft/s or $v(5) = s'(5) = \frac{220}{3}$ ft/s. Find $a$. But $s'(t) = \int a\,dt = at + c$. Since $s'(0) = a \cdot 0 + c = 44 \Rightarrow c = 44$. So $s'(t) = at + 44$. So $s'(5) = 5a + 44 = \frac{220}{3} \Rightarrow 5a = \frac{220}{3} - 44 = \frac{88}{3}$. Thus $a = \frac{88}{15}$.

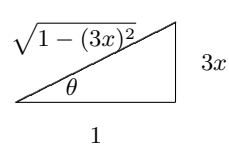
**35. a)** Given: $a(t) = s''(t) = -32$ ft/s$^2$. $v(0) = s'(0) = 0$ ft/s. Find $s(0)$. We know $s(5) = 0$. $s'(t) = \int -32\,dt = -32t + c$. But $s'(0) = a \cdot 0 + c = 0$ so $c = 0$ and $s'(t) = -32t$ Next $s(t) = \int -32t\,dt = -16t^2 + c$. But $s(5) = -16(5)^2 + c = 0 \Rightarrow c = 400$. So $s(t) = -16t^2 + 400$ and $s(0) = 400$ ft.

**36.** Draw an appropriate right triangle to help evaluate each expression.

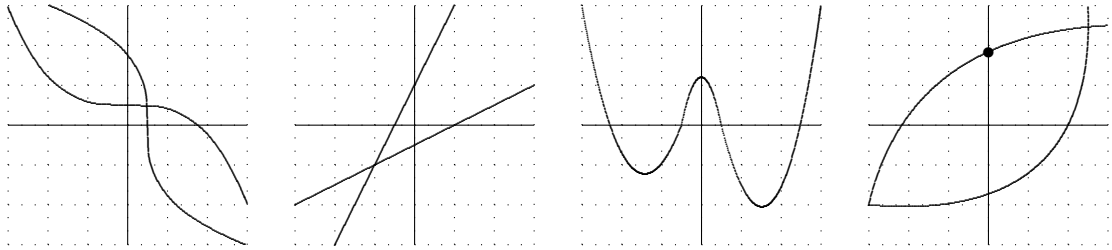
**a)** $\sin(\arccos 4x) = \sin\theta = \sqrt{1 - 16x^2}$




**b)** $\sec(\arctan 3x) = \sec\theta = \dfrac{\sqrt{1 - 9x^2}}{1} = \sqrt{1 - 9x^2}$



**37.** Only (c) is not one-to-one. So the others have inverses.



**38.** Solutions:

**a)** $-27\cos(x/3) + c$

**b)** $= 6\displaystyle\int \frac{1}{25+x^2}\,dx = \frac{6}{5}\arctan\frac{x}{5} + c$

**c)** $= 6\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx = 6\arcsin\frac{x}{5} + c$

**d)** $\displaystyle\int (x^{2/3} + x^{3/2})\,dx = \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + c$

**e)** $\dfrac{3}{8}\sec(8x) + c$

**f)** $\displaystyle\int \frac{1}{x} + x^{-1/2}\,dx = \ln|x| + 2x^{1/2} + c$

**g)** $\dfrac{1}{2}\displaystyle\int (e^{2x} - e^{-2x})\,dx = \frac{1}{4}(e^{2x} + e^{-2x}) + c$

**39.** Implicit differentiation:

**a)** $\dfrac{dy}{dx} = \dfrac{x}{y}$

**b)** $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{e^y}$

**c)** $\dfrac{dy}{dx} = \dfrac{2x+1}{3+\frac{1}{y}}$

**d)** $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{\sec y \tan y}$

**e)** $\dfrac{dy}{dx} = -\dfrac{2x+2xy^2}{2x^2y+3y^2}$

**f)** $\dfrac{dy}{dx} = \dfrac{-y^2}{2xy}$

**g)** $\dfrac{dy}{dx} = \dfrac{2-\cos(x+y)}{\cos(x+y)}$

**h)** $m = \left.\frac{dy}{dx}\right|_{(2,-3)} = \frac{-9}{-12} = \frac{3}{4}$. So $y + 3 = \frac{3}{4}(x-2)$ or $y = \frac{3}{4}x - \frac{9}{2}$