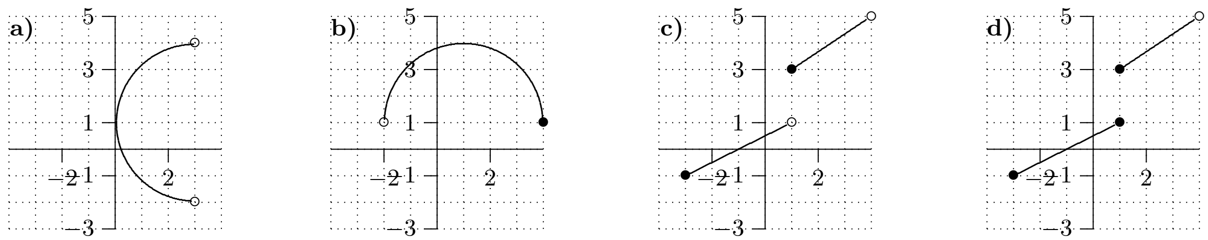
Math 130 Lab 1: Review and Difference Quotients

You may want a 3-ring binder for your lab work. Some problems, especially on today's lab, can be worked out on these sheets. Others will require work on a separate page. Keep this work for future study.

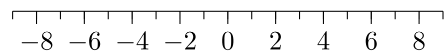
1. Determine which of the following are graphs of functions. Explain your answer.



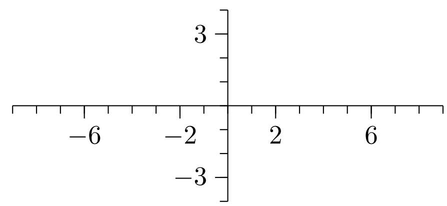
2. Absolute values will be important when we carefully define limits. Solve each of the following.

a) $|x-5| \le 2$ **b)** $|y-2| < 1$

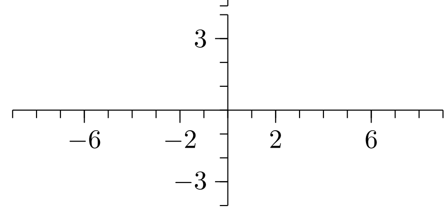
c) Now graph the solution set to part (a) on the given number line.



d) Graph all the points (x, y) in the *plane* which satisfy (a).



e) Graph all points in the plane that satisfy *both* (a) and (b).



3. Exponents. Simplify each of these expressions writing a single power of x.

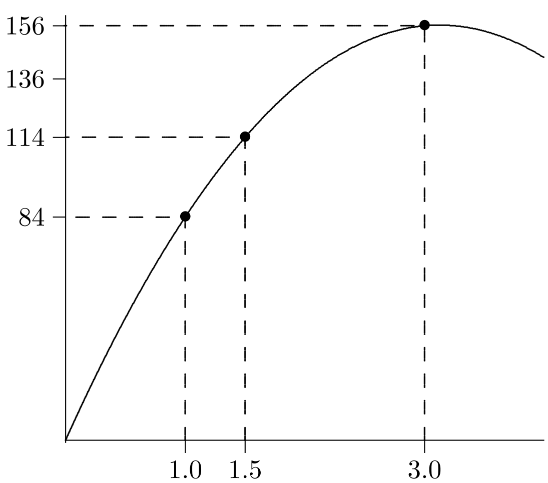
a) $4\sqrt[3]{x^5}$ **b)** $\dfrac{\sqrt{x^8}}{2x^2}$ **c)** $\dfrac{x^3x^{-5}}{2x^6x^{-1}}$ **d)** $\left(\dfrac{x^{3/5}}{x^{-2}}\right)^4$

4. Lines and slopes will be critical to what we do at the beginning of the course.

a) Find the equation of the line through $(2, -7)$ and $(-3, 5)$.

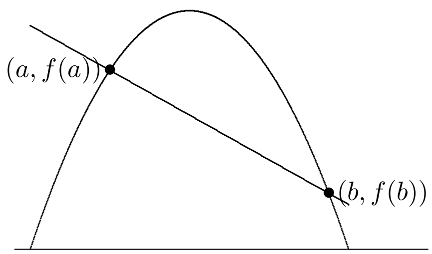
b) Find the equation of the line through $(2, -4)$ with slope 3.

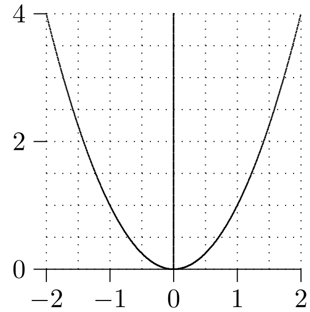
5. Average Velocity. The graph below gives the position (in meters) of an object moving along a line at time t, over a 2-second interval. Find the average velocity of the object (use the difference quotient) for the given intervals.



Time interval	Average velocity
$[1, 3]$	
$[1, 1.5]$	

6. a) A **secant line** to the graph of a function $y = f(x)$ is a line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the curve (see below, left). What is the general formula for the **slope** m_{sec} of the secant line containing the points $(b, f(b))$ and $(a, f(a))$. This general formula for m_{sec} works for all functions. [Your answer will involve, $a, b, f(a), f(b)$.]





Gen'l Formula: $m_{\text{sec}} =$

b) Suppose we want secant lines for $f(x) = x^2$. So what are $f(a)$ and $f(b)$ now? Substitute these values for $f(a)$ and $f(b)$ in your general formula for m_{sec}. Simplify the expression by factoring. You should now have a very simple formula for the slope of the secant, m_{sec}, between any two points on the curve $y = x^2$. **Use this formula in the next two parts.** Formula: m_{sec}=____________________

c) Still assume that $f(x) = x^2$. Now let $a = -1$ and $b = 2$. Substitute these particular values in your final formula for m_{sec} in part (b) to find the slope m of the secant line through $(-1, f(-1))$ and $(2, f(2))$. Now that you know the slope what is the equation of the line? Draw this line on the axes above.

d) Again use your formula to quickly find the slope m_{sec} for the line through $(-1, f(-1))$ and $(0, f(0))$.

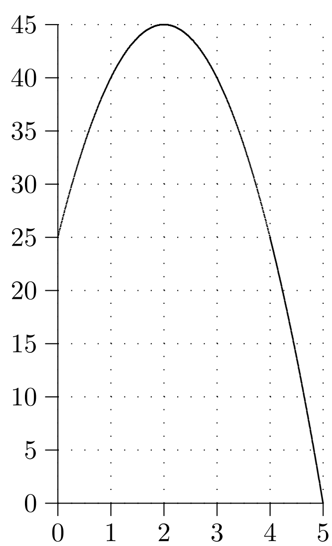
e) If the secant line through $(-1, f(-1))$ and $(b, f(b))$ has slope 17, what is b?

7. Average velocity is 'change in position over change in time.' When position is given by a function $y = f(x)$, where x is time, then average velocity is just 'change in y over change in x' which is just the secant slope, m_{sec}, or difference quotient again. Assume a ball is thrown upward from the roof of Gulick Hall so that its position above the ground after x seconds is given by $y = f(x) = -5x^2 + 20x + 25$ meters over the time interval $0 \le x \le 5$ (see graph below).

a) Let $(x, f(x))$ be any point on the curve. Find the general formula for the average velocity between $(3, f(3))$ and $(x, f(x))$. Remember: This is the same thing as finding m_{sec} between these two points using 3 and x instead of a and b. Simplify your answer. Hint: First factor out -5 in the numerator.

Simplified Formula for $m_{\text{sec}} =$

b) Use your formula from part (a) to evaluate average velocities in the table below. This very, very easy.



Time interval $[3, x]$	Average velocity on $[3, x]$
$[3, 3.1]$	
$[3, 3.01]$	
$[3, 3.001]$	
$[3, 3.0001]$	
$[3, 3]$	*!*&*%**%$***
$[2.9999, 3]$	
$[2.999, 3]$	
$[2.99, 3]$	
$[2.9, 3]$	

c) Why is "*!*&*%**%$***" in the table above?

d) Make a conjecture (educated guess) of the instantaneous velocity right at time $x = 3$: ____________________

8. **Yet another way to calculate m_{sec}.** Slope is simply the change in y over the change in x. This is easy to compute for a straight line. But in calculus we want to determine the 'slope' of a curve $y = f(x)$. Suppose that h represents the change in x, i.e., x changes from x to $x + h$ (instead of a to b). Then $y = f(x)$ changes from $f(x)$ to $f(x + h)$. So the slope becomes

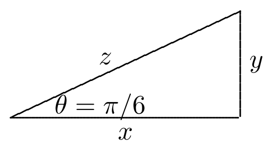
$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}.$$

This last expression is called the **difference quotient**. Care must be taken in computing and simplifying $f(x+h) - f(x)$. For example, if $f(x) = x^2 + 1$ then

$$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 + 1] - [x^2 + 1]}{h} = \frac{x^2 + 2xh + h^2 + 1 - [x^2 + 1]}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

Notice how m_{sec} simplifies!! You should be comfortable with this algebra. Compute **difference quotient** $\dfrac{f(x+h) - f(x)}{h}$ for

a) $f(x) = 3x^2 + 2$ **b)** $f(x) = \dfrac{2}{x}$ **c)** $f(x) = x^2 - x$



9. A right triangle has sides as indicated.

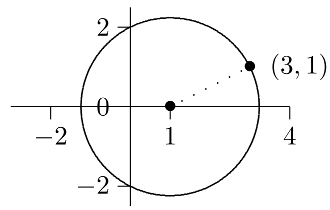
a) If $z = 20$, solve for x. (Use a trig function.)

b) If $z = 10$, solve for y.

c) Instead, if $x = 12$, solve for y.

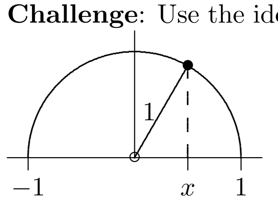
d) An airplane leaves the runway climbing at an angle of 18° with a speed of 275 feet per second (see figure). Find the altitude of the plane after 1 minute. (Reset your calculator to radians.)

10. **a)** **Fun.** Find the equation of the tangent line to the circle at the point indicated. Hint: How is the tangent line related to the radius?



b) Circles don't have slopes in the usual sense. However, how should you "define" the "slope" of the circle at the point $(3, 1)$?

11. **Challenge**: Use the ideas in previous problem. A **unit** semi-circle is drawn below and a general point is marked on it.



a) Determine a formula for "slope" of the circle at the point indicated. (You will first need to find the y-coordinate of the point • on the circle in terms of x.)

b) Use your formula to calculate the slope of the circle when $x = 0.9$ and when $x = -0.4$.

c) What is the domain of your "slope" function?

d) Use your formula to determine when the slope is 0. Does that make sense given the picture?

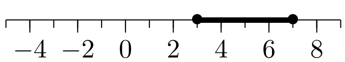
e) Use your formula to determine when the slope is positive. Negative. Compare to the graph. Express your answers using interval notation.

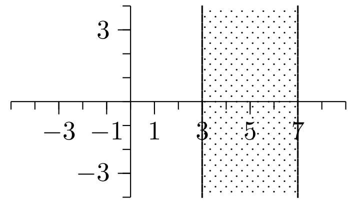
Math 130 Lab 1 Brief Answers

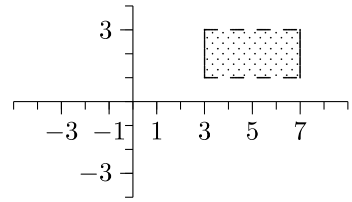
Complete detailed answers will be online at our website. Ask me or a TA for help.

1. **(a)** and **(d)** fail the vertical line test and are **not** functions.

2. **a)** $-2 \leq x - 5 \leq 2$ or $3 \leq x \leq 7$.
b) $-1 < y - 2 < 1$ or $1 < y < 3$.

c) 

d) 

e) 

3. **a)** $4x^{5/3}$ **(b)** $\frac{x^2}{2}$ **(c)** $\frac{x^{-7}}{2}$ **(d)** $x^{52/5}$

4. **a)** $y = -\frac{12}{5}x - \frac{11}{5}$. **(b)** $y = 3x - 10$.

5. The average velocity is the difference quotient: $\frac{\Delta\text{Position}}{\Delta\text{time}}$. We get 36 m/s and 60 m/s

6. **a)** The slope is the difference quotient: $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a}$.
b) Simplifies to $m_{\text{sec}} = b + a$, where $b \neq a$.
c) With $b = 2$ and $a = 1$, the slope will be $m_{\text{sec}} = b + a = 2 + (-1) = 1$. The line: $y = x + 2$.
d) With $b = 0$. The slope will be $m_{\text{sec}} = b + a = 0 - 1 = -1$. Line: $y = -x$
e) $a = -1$ so $m_{\text{sec}} = b - a = b - 1 = 17$, so $b = 18$.

7. **a)** Simplifies to Ave Vel $= \dfrac{f(b) - f(3)}{b-3} \dfrac{-5(b-3)(b-1)}{b-3} = -5(b-1)$

Time interval$[3, b]$	Average velocity on $[3, b]$
$[3, 3.1]$	-10.5
$[3, 3.01]$	-10.05
$[3, 3.001]$	-10.005
$[3, 3.0001]$	-10.0005

Time interval$[3, b]$	Average velocity on $[3, b]$
$[2.9999, 3]$	-9.9995
$[2.999, 3]$	-9.995
$[2.99, 3]$	-9.95
$[2.9, 3]$	-9.5

c) Because $\frac{f(b)-f(3)}{b-3}$ is not defined at $b = 3$. (Division by 0.)
d) Estimate from the table: the average velocity gets close to -10 m/s as b gets close to 3.

8. **a)** $6x + 3h$. **(b)** $\frac{-2}{x(x+h)}$.

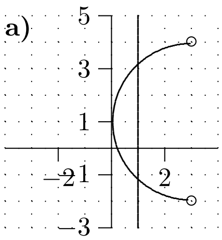
9. **a)** $\cos(\pi/6) = \frac{x}{z}$. So $\frac{\sqrt{3}}{2} = \frac{x}{20}$. Thus $x = 10\sqrt{3}$.
b) $\sin(\pi/6) = \frac{x}{z}$. So $\frac{1}{2} = \frac{y}{10}$. And $y = 5$.
c) $\tan(\pi/6) = \frac{\sqrt{3}}{3} = \frac{y}{12}$. So $y = 4\sqrt{3}$.
d) The hypotenuse is $60 \times 275 = 16500$ feet. So $\sin(18) = \frac{a}{16500}$. So $a = 16500 \sin(18) \approx 5098.8$ feet.

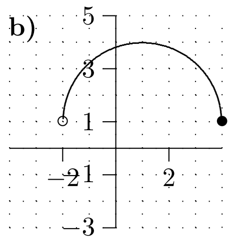
10. **a)** The tangent is perpendicular to the radius through $(1, 0)$ and $(3, 1)$. This radial line has equation $y = \frac{1}{2}x - \frac{1}{2}$. Using negative reciprocals, the slope of the tangent is $m = -2$. The tangent passes through the point $(3, 1)$, so the equation of the tangent is $y - 1 = -2(x - 3)$ or $y = -2x + 7$.
b) The "slope" of the circle at $(3, 1)$ is the same as the slope of its tangent line there: $m = -2$.

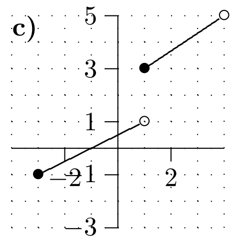
11. **a)** The slope of the tangent is $m_{\tan} = -\dfrac{x}{\sqrt{1 - x^2}}$.

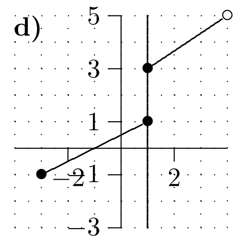
Math 130 Lab 1 Answers

1. **(a)** and **(d)** fail the vertical line test and are **not** functions.



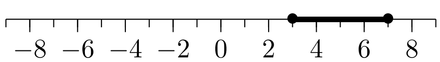


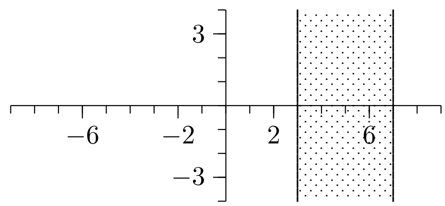




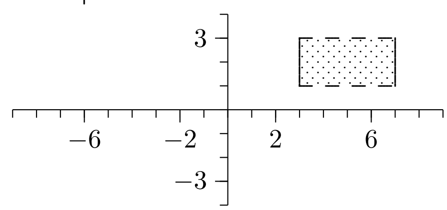
2. **a)** $|x-5| \le 2$ means $-2 \le x-5 \le 2$ or $3 \le x \le 7$.

b) $|y-2| < 1$ means $-1 < y-2 < 1$ or $1 < y < 3$.

c) 

d) 

e) We need $3 \le x \le 7$ and $1 < y < 3$:



3. Exponents.

a) $4\sqrt[3]{x^5} = 4x^{5/3}$

b) $\dfrac{\sqrt{x^8}}{2x^2} = \dfrac{x^4}{2x^2} = \dfrac{x^2}{2}$

c) $\dfrac{x^3x^{-5}}{2x^6x^{-1}} = \dfrac{x^{-2}}{2x^5} = \dfrac{x^{-7}}{2}$

d) $\left(\dfrac{x^{3/5}}{x^{-2}}\right)^4 = (x^{13/5})^4 = x^{52/5}$

4. **a)** $m = \frac{-7-5}{2-(-3)} = -\frac{12}{5}$. Eqn: $y-(-7) = -\frac{12}{5}(x-2)$ or $y = -\frac{12}{5}x - \frac{11}{5}$.

b) $y-(-4) = 3(x-2)$ or $y = 3x-10$.

5. The average velocity is the difference quotient: $\frac{\Delta\text{Position}}{\Delta\text{time}}$. We get

$$\frac{156-84}{3-1} = 36 \text{ m/s} \qquad \text{and} \qquad \frac{114-84}{1.5-1} = 60.$$

6. **a)** The slope is the difference quotient: $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a}$.

b) We get $m_{\text{sec}} = \frac{f(b)-f(a)}{b-a} = \frac{b^2-a^2}{b-a} = \frac{(b-a)(b+a)}{b-a} = b+a$, where $b \ne a$.

c) With $b=2$ and $a=1$, the slope will be $m_{\text{sec}} = b+a = 2+(-1) = 1$. The line: $y = x+2$.

d) With $b=0$. The slope will be $m_{\text{sec}} = b+a = 0-1 = -1$. Line: $y=-x$

e) $a=-1$ so $m_{\text{sec}} = b-a = b-1 = 17$, so $b=18$.

7. **a)** Ave Vel $= \dfrac{f(b)-f(3)}{b-3} = \dfrac{-5b^2+20b+25-40}{b-3} = \dfrac{-5b^2+20b-15}{b-3}$

$$= \frac{-5(b^2-4b+3)}{b-3} = \frac{-5(b-3)(b-1)}{b-3} = -5(b-1)$$

b) Use the formula that you just found to evaluate average velocities.

Time interval$[3, b]$	Average velocity on $[3, b]$
$[3, 3.1]$	-10.5
$[3, 3.01]$	-10.05
$[3, 3.001]$	-10.005
$[3, 3.0001]$	-10.0005

Time interval$[3, b]$	Average velocity on $[3, b]$
$[2.9999, 3]$	-9.9995
$[2.999, 3]$	-9.995
$[2.99, 3]$	-9.95
$[2.9, 3]$	-9.5

c) Because $\frac{f(b)-f(3)}{b-3}$ is not defined at $b = 3$. (Division by 0.)

d) Estimate from the table: the average velocity gets close to -10 m/s as b gets close to 3.

8. Be very careful about losing signs in the numerator.

a) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3(x+h)^2+2-[3x^2+2]}{h} = \dfrac{3(x^2+2xh+h^2)+2-3x^2-2}{h} = \dfrac{3x^2+6xh+3h^2-3x^2}{h} = \dfrac{6xh+3h^2}{h} =$ $6x+3h$.

b) $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\frac{2}{x+h}-\frac{2}{x}}{h} = \dfrac{\frac{2x-2x-2h}{x(x+h)}}{h} = \dfrac{-2h}{x(x+h)h} = \dfrac{-2}{x(x+h)}$.

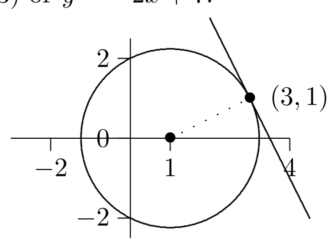
9. a) $\cos(\pi/6) = \frac{x}{z}$. So $\frac{\sqrt{3}}{2} = \frac{x}{20}$. Thus $x = 10\sqrt{3}$.

b) $\sin(\pi/6) = \frac{x}{z}$. So $\frac{1}{2} = \frac{y}{10}$. And $y = 5$.

c) $\tan(\pi/6) = \frac{\sqrt{3}}{3} = \frac{y}{12}$. So $y = 4\sqrt{3}$.

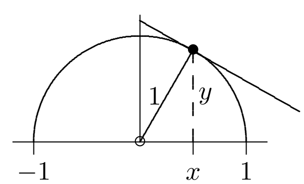
d) The hypotenuse is $60 \times 275 = 16500$ feet. So $\sin(18) = \frac{a}{16500}$. So $a = 16500\sin(18) \approx 5098.8$ feet.

10. a) The tangent is perpendicular to the radius through $(1, 0)$ and $(3, 1)$. This radial line has equation $y = \frac{1}{2}x - \frac{1}{2}$. Using negative reciprocals, the slope of the tangent is $m = -2$. The tangent passes through the point $(3, 1)$, so the equation of the tangent is $y - 1 = -2(x-3)$ or $y = -2x + 7$.



b) The "slope" of the circle at $(3, 1)$ is the same as the slope of its tangent line there: $m = -2$.

11. a) Using a right triangle we get: $x^2 + y^2 = 1$ so $y = \sqrt{1-x^2}$. The tangent is perpendicular to the radius through $(0, 0)$ and (x, y). The radial line has slope $m_{\text{rad}} = \dfrac{y-0}{x-0} = \dfrac{\sqrt{1-x^2}}{x}$. The slope of the tangent is the negative reciprocal or $m_{\tan} = -\dfrac{x}{\sqrt{1-x^2}}$.



b) When $x = 0.9$, then $m_{\tan} = -\frac{x}{\sqrt{1-x^2}} = -\frac{0.9}{\sqrt{1-0.81}} \approx 2.0647$ and when $x = -0.4$, then $m_{\tan} = -\frac{x}{\sqrt{1-x^2}} = \frac{0.4}{\sqrt{1-0.16}} \approx$ 0.4364.

c) The domain of $m_{\tan} = -\frac{x}{\sqrt{1-x^2}}$ is $-1 < x < 1$.

d) The slope is 0 at $x = 0$ because the tangent line is horizontal at the top of the semi-circle.

e) Positive (uphill): $-1 < x < 0$. Negative (downhill): $0 < x < 1$.