

Math 130: Lab 2

☞ **Good limit grammar.** Remember to use equal signs and limit symbols in your limit calculations. When numerically evaluating the limit in the last step(s), do not use the limit symbol because you have found the limit. For example,

$$\lim_{x \rightarrow -4} \sqrt[3]{3x - 15} \stackrel{\text{FracPow}}{=} \sqrt[3]{\lim_{x \rightarrow -4} 3x - 15} \stackrel{\text{Poly}}{=} \sqrt[3]{-27} = -3.$$

1. Assume $\lim_{x \rightarrow 1} f(x) = 4$ and $\lim_{x \rightarrow 1} g(x) = -3$. Evaluate these limits and indicate the properties that you used. E.g.,

$$\lim_{x \rightarrow 1} f(x)g(x) + 7 \stackrel{\text{Sum=Rule}}{=} \lim_{x \rightarrow 1} (f(x)g(x)) + \lim_{x \rightarrow 1} 7 \stackrel{\text{Product}}{=} \lim_{x \rightarrow 1} f(x) \lim_{x \rightarrow 1} g(x) + \lim_{x \rightarrow 1} 7 \stackrel{\text{Given, Constant}}{=} 4(-3) + 7 = -5.$$

a) $\lim_{x \rightarrow 1} 2g(x) - 10\sqrt{x}$ b) $\lim_{x \rightarrow 1} x^3 g(x) + \frac{g(x)}{f(x)}$ c) $\lim_{x \rightarrow 1} \sqrt{9f(x)} + \frac{x+1}{x^2+1}$

2. Determine which of these limits are indeterminate (that, is are of the form $\frac{0}{0}$) and so require ‘more work.’ Evaluate each of the limits using basic limit properties and any required algebraic simplifications. Make a note of which properties or theorems you applied, e.g., “constant multiple” or “polynomial”.

a) $\lim_{x \rightarrow -2} 2x^2 - 3x - 1$ b) $\lim_{x \rightarrow 3} \frac{8x^2}{x^2 - 3}$ c) $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 3x}$ d) $\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x^2 - 25}$

e) $\lim_{x \rightarrow -1} \frac{x^3 - x}{x^2 - 5x - 6}$ f) $\lim_{x \rightarrow 0} \frac{\frac{2}{x+4} - \frac{1}{2}}{x}$ g) $\lim_{x \rightarrow 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{2 - x}$ Do Later?

h) $\lim_{x \rightarrow 3} \frac{x+1}{\sqrt[3]{x-11}}$ i) $\lim_{x \rightarrow 6} \frac{\sqrt{x} - \sqrt{6}}{x - 6}$ j) $\lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x+1} - 2}$ Do Later?

3. a) **Like a HW Problem.** A rock is thrown up from a canyon 100 ft high on the planet Pluto. The height of the ball above the ground after t seconds is given by $s(t) = -2t^2 + 10t + 100$ ft. The goal is to find the instantaneous velocity right at time $t = 5$. Remember that

$$\text{Inst Vel} = \lim_{t \rightarrow 5} \text{Ave Vel} = \lim_{t \rightarrow 5} \text{Difference Quotient} = \lim_{t \rightarrow 5} \frac{s(t) - s(5)}{t - 5}.$$

Is this limit indeterminate? Evaluate the limit—what was the velocity?

- b) What is the slope of the curve $y = s(t)$ right at $t = 5$? (This should be easy and require no new work.)
c) What is the equation of the tangent line to the curve at $t = 5$? [The tangent line goes through the point on the curve where $t = 5$ and has the same slope as the curve.]

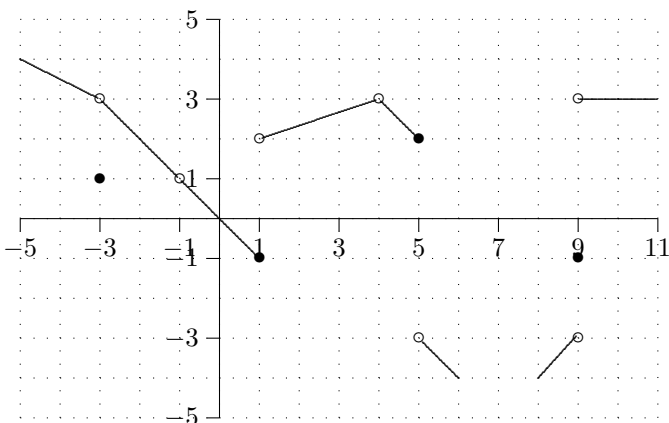
4. Let $f(x) = \begin{cases} 1 - x, & \text{if } x < 2 \\ 4 & \text{if } x = 2. \\ \frac{x^2}{x+2} & \text{if } x > 2 \end{cases}$. Determine the following limits, if they exist. Careful: Make sure to use the appropriate function rule for each limit. Do you approach with values greater or smaller than 2?

a) $\lim_{x \rightarrow 2^-} f(x)$ b) $\lim_{x \rightarrow 2^+} f(x)$ c) $\lim_{x \rightarrow 2} f(x)$ d) $\lim_{x \rightarrow 4^-} f(x)$

5. Let $f(x) = \begin{cases} \frac{mx+2}{2x-1}, & \text{if } x < 1 \\ 4 & \text{if } x = 1. \\ \frac{m^2 x^2}{4x-3} & \text{if } x > 1 \end{cases}$. Determine all values for m for which $\lim_{x \rightarrow 1} f(x)$ exists.

☞: Some answers. #1: $-16, -\frac{15}{4}, 7$. #2: $13, 12, 2, \frac{7}{10}, -\frac{2}{7}, -\frac{1}{8}, \frac{1}{4}, -2, \frac{1}{2\sqrt{6}}, 4$. #3: $-10, -10, y = -10t + 150$. #4: $-1, 1, \text{DNE}, \frac{8}{3}$. #5: $m = -1, 2$. #7. P: e, g. R: a, d. #8: $-4x + 10, -\frac{3}{x^2}, -\frac{1}{2}x^{-3/2}$. #9: $1, 1, 1; 1, 2, \text{DNE}, \text{DNE} -2$.

6. a) Fill in the first four columns below by using the graph to determine the limits, if they exist.
- b) A function f is **continuous** at a if $\lim_{x \rightarrow a} f(x) = f(a)$, i.e., if the function's value and its limit both exist and are the same number. Fill in $f(a)$ and then determine whether f is continuous (Cont) there (Y /N).
- c) Complete the graph between 6 and 8 so that $f(7) = -5$ but $\lim_{x \rightarrow 7} f(x) = -3$.



a	$\lim_{x \rightarrow a^-} f(x)$	$\lim_{x \rightarrow a^+} f(x)$	$\lim_{x \rightarrow a} f(x)$	$f(a)$	Cont
-4					
-3					
-1					
0					
1					
4					
5					
9					

7. Use the most appropriate description for each: (P) polynomial; (R) rational function; (N) neither. Mark each.

a) $\frac{2x-4}{x^2} + \frac{1}{x}$

b) $\sqrt{\frac{x^2+12}{3x^2+1}}$

c) $\frac{x}{\sqrt{x^2+1}}$

d) $\frac{2x+1}{(x-4)(x+1)}$

e) $x^2(2x-5)(x+3)$

f) $\arcsin(x^2+3x+1)$

g) $\frac{21}{2}x^{11} - 9.1x^4 + \pi$

h) $x^{-1/2} + 1$

8. Here's another way to compute the difference quotient (i.e., average velocity or secant slope). Instead of using two points x and a , we sometimes use x and $x+h$, where we think of h as a small number (or small amount of time that has passed). Then

$$\text{Difference Quotient} = \frac{f(x+h) - f(x)}{(x+h) - h} \text{ which simplifies to } \frac{f(x+h) - f(x)}{h}.$$

Evaluate $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ for each of the following functions.

a) $f(x) = -2x^2 + 10x + 100$

b) $f(x) = \frac{3}{x}$

c) **Challenge**: $f(x) = \frac{1}{\sqrt{x}}$

9. The **floor function** is the function denoted by $f(x) = \lfloor x \rfloor$ and is defined by $\lfloor x \rfloor$ = the greatest integer less than or equal to x . For example, $\lfloor 2.1 \rfloor = 2$, and $\lfloor 2.9999 \rfloor = 2$, and $\lfloor -2.1 \rfloor = -3$, and $\lfloor 1 \rfloor = 1$. Make a plot of the graph of $\lfloor x \rfloor$. Then determine the following limits.

a) $\lim_{x \rightarrow 1.4^-} \lfloor x \rfloor$

b) $\lim_{x \rightarrow 1.4^+} \lfloor x \rfloor$

c) $\lim_{x \rightarrow 1.4} \lfloor x \rfloor$

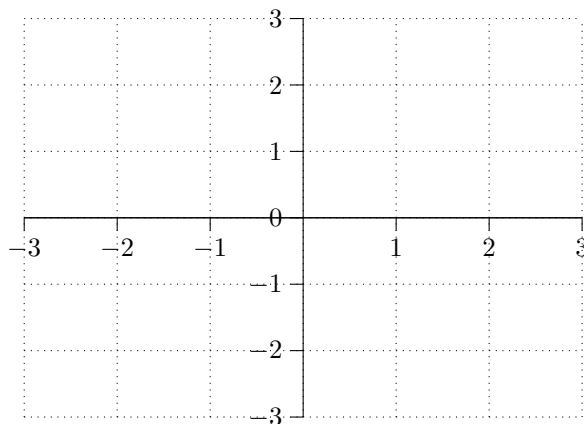
d) $\lim_{x \rightarrow 2^-} \lfloor x \rfloor$

e) $\lim_{x \rightarrow 2^+} \lfloor x \rfloor$

f) $\lim_{x \rightarrow 2} \lfloor x \rfloor$

g) $\lim_{x \rightarrow 0} \lfloor x \rfloor$

h) $\lim_{x \rightarrow -1.5} \lfloor x \rfloor$



Math 130: Lab 2 Answers

1. a) $\lim_{x \rightarrow 1} 2g(x) - 10\sqrt{x} \stackrel{\text{Diff, Const mult}}{=} 2 \lim_{x \rightarrow 1} g(x) - 10 \lim_{x \rightarrow 1} \sqrt{x} \stackrel{\text{Frac}=\text{Pow}}{=} 2(-3) - 10\sqrt{1} = -16.$
- b) $\lim_{x \rightarrow 1} x^3 g(x) + \frac{g(x)}{f(x)} \stackrel{\text{Sum}}{=} \lim_{x \rightarrow 1} x^3 g(x) + \lim_{x \rightarrow 1} \frac{g(x)}{f(x)} \stackrel{\text{Prod}=\text{Quot}}{=} \lim_{x \rightarrow 1} x^3 \cdot \lim_{x \rightarrow 1} g(x) + \frac{\lim_{x \rightarrow 1} g(x)}{\lim_{x \rightarrow 1} f(x)} \stackrel{\text{Poly}}{=} 1^3 \cdot (-3) + \frac{-3}{4} = -\frac{15}{4}$
- c) $\lim_{x \rightarrow 1} \sqrt{9f(x)} + \frac{x+1}{x^2+1} \stackrel{\text{Sum}}{=} \lim_{x \rightarrow 1} \sqrt{9f(x)} + \lim_{x \rightarrow 1} \frac{x+1}{x^2+1} \stackrel{\text{Root}}{=} \sqrt{\lim_{x \rightarrow 1} 9f(x)} + \lim_{x \rightarrow 1} \frac{x+1}{x^2+1} \stackrel{\text{Const}=\text{Mult}}{=} \sqrt{9 \lim_{x \rightarrow 1} f(x)} + \lim_{x \rightarrow 1} \frac{x+1}{x^2+1} \stackrel{\text{Rat'lfcn}}{=} \sqrt{9(4)} + \frac{2}{2} = 7.$
2. All are indeterminate except as noted.
- a) Not indeterminate. Easy: $\lim_{x \rightarrow -2} 2x^2 - 3x - 1 \stackrel{\text{Poly}}{=} 8 + 6 - 1 = 13.$
- b) Not indeterminate. $\lim_{x \rightarrow 3} \frac{8x^2}{x^2 - 3} \stackrel{\text{Rat'l}}{=} \frac{72}{6} = 12.$
- c) Factor. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 3x} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)x} = \lim_{x \rightarrow 3} \frac{x+3}{x} \stackrel{\text{Rat'l}}{=} 2.$
- d) Factor. $\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x^2 - 25} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 5} \frac{(x+2)(x-5)}{(x-5)(x+5)} = \lim_{x \rightarrow 5} \frac{x+2}{x+5} \stackrel{\text{Rat'l}}{=} \frac{7}{10}.$
- e) Factor. $\lim_{x \rightarrow -1} \frac{x^3 - x}{x^2 - 5x - 6} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{(x+1)(x-6)} = \lim_{x \rightarrow -1} \frac{x(x-1)(x+1)}{(x+1)(x-6)} = \lim_{x \rightarrow -1} \frac{x(x-1)}{x-6} \stackrel{\text{Rat'l}}{=} -\frac{2}{7}.$
- f) Compound Frac. $\lim_{x \rightarrow 0} \frac{\frac{2}{x+4} - \frac{1}{2}}{x} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 0} \frac{\frac{4-(x+4)}{2(x+4)}}{x} = \lim_{x \rightarrow 0} \frac{x}{2(x+4)x} = \lim_{x \rightarrow 0} \frac{1}{2(x+4)} \stackrel{\text{Rat'l}}{=} -\frac{1}{8}.$
- g) Comp'd Frac. $\lim_{x \rightarrow 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{2-x} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 2} \frac{\frac{4-x^2}{4x^2}}{2-x} = \lim_{x \rightarrow 2} \frac{4-x^2}{(4x^2)(2-x)} = \lim_{x \rightarrow 2} \frac{(2-x)(2+x)}{(4x^2)(2-x)} = \lim_{x \rightarrow 2} \frac{2+x}{4x^2} \stackrel{\text{Rat'l}}{=} \frac{4}{16} = \frac{1}{4}.$
- h) Not indeterminate. Use the quotient, poly, difference, fractional powers rules: $\lim_{x \rightarrow 3} \frac{x+1}{\sqrt[3]{x-11}} = \frac{4}{\sqrt[3]{-8}} = -2.$
- i) Conjugates. $\lim_{x \rightarrow 6} \frac{\sqrt{x} - \sqrt{6}}{x-6} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 6} \frac{\sqrt{x} - \sqrt{6}}{x-6} \cdot \frac{\sqrt{x} + \sqrt{6}}{\sqrt{x} + \sqrt{6}} = \lim_{x \rightarrow 6} \frac{x-6}{(x-6) \cdot (\sqrt{x} + \sqrt{6})} = \lim_{x \rightarrow 6} \frac{1}{\sqrt{x} + \sqrt{6}} \stackrel{\text{FracPwr, Quot}}{=} \frac{1}{2\sqrt{6}}.$
- j) Conjugates. $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1} - 2} \stackrel{\nearrow 0}{=} \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x+1} - 2} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1} + 2)}{x+1-4} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x+1} + 2)}{x-3} = \lim_{x \rightarrow 3} \sqrt{x+1} + 2 \stackrel{\text{Root, Poly, Sum}}{=} 4.$
3. a) $\text{Inst Vel} = \lim_{t \rightarrow 5} \text{Ave Vel} = \frac{s(t) - s(5)}{t - 5} = \lim_{t \rightarrow 5} \frac{-2t^2 + 10t + 100 - [-50 + 50 + 100]}{t - 5}$
 $= \lim_{t \rightarrow 5} \frac{-2t^2 + 10t}{t - 5} = \lim_{t \rightarrow 5} \frac{-2t(t - 5)}{t - 5} = \lim_{t \rightarrow 5} -2t \stackrel{\text{Poly}}{=} -10 \text{ ft/s}$
- b) The 'slope of the curve' or tangent slope is the same as the instantaneous velocity: $-10.$
- c) The slope is -10 and the line goes through $(5, 100)$. So the tangent line is $y - 100 = -10(t - 5)$ or $y = -10t + 150.$
4. Use the appropriate function definition
- a) We approach 2 from the left ($x < 2$), so $f(x) = 1 - x$. So $\lim_{x \rightarrow 2^-} f(x) \stackrel{x \leq 2}{=} \lim_{x \rightarrow 2^-} 1 - x \stackrel{\text{Poly(Linear)}}{=} -1.$
- b) We approach 2 from the right ($x > 2$), so $f(x) = \frac{x^2}{x+2}$. So $\lim_{x \rightarrow 2^+} f(x) \stackrel{x \geq 2}{=} \lim_{x \rightarrow 2^+} \frac{x^2}{x+2} \stackrel{\text{Rational}}{=} \lim_{x \rightarrow 2^+} x = \frac{4}{4} = 1.$
- c) $\lim_{x \rightarrow 2} f(x)$ DNE because $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x).$
- d) Now x is near but smaller than 4, so $x > 2$. Thus $f(x) = \frac{x^2}{x+2}$. So $\lim_{x \rightarrow 4^-} f(x) \stackrel{x \leq 4}{=} \lim_{x \rightarrow 4^-} \frac{x^2}{x+2} \stackrel{\text{Rat'l}}{=} \frac{16}{6} = \frac{8}{3}.$

5. This is a piecewise function and split at $x = 1$. Calculate the left and right limits and compare their values.

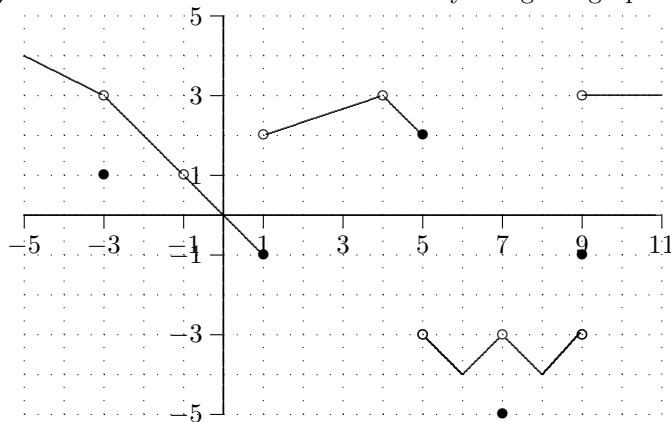
a) Approach 1 from the left ($x < 1$), so $f(x) = \frac{mx+2}{2x-1}$. So $\lim_{x \rightarrow 1^-} f(x) \stackrel{x \leq 1}{=} \lim_{x \rightarrow 1^-} \frac{mx+2}{2x-1} \stackrel{\text{Rational}}{=} m+2$.

b) We approach 1 from the right ($x > 1$), so $f(x) = \frac{m^2x^2}{4x-3}$. So $\lim_{x \rightarrow 1^+} f(x) \stackrel{x \geq 1}{=} \lim_{x \rightarrow 1^+} \frac{m^2x^2}{4x-3} \stackrel{\text{Rational}}{=} m^2$. For the two-sided limit to exist the one-sided limits must be equal, so $m^2 = m+2$ or $m^2 - m - 2 = (m-2)(m+1) = 0$. So $m = 2$ or -1 .

c) $\lim_{x \rightarrow 2} f(x)$ DNE because $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$.

d) Now x is near but smaller than 4, so $x > 2$. Thus $f(x) = \frac{x^2}{x+2}$. So $\lim_{x \rightarrow 4^-} \frac{x^2}{x+2} \stackrel{x \leq 4}{=} \lim_{x \rightarrow 4^-} \frac{x^2}{x+2} \stackrel{\text{Rat}'1}{=} \frac{16}{6} = \frac{8}{3}$.

6. a) Fill in the first four columns below by using the graph to determine the limits, if they exist.



a	$\lim_{x \rightarrow a^-} f(x)$	$\lim_{x \rightarrow a^+} f(x)$	$\lim_{x \rightarrow a} f(x)$	$f(a)$	Cont
-4	3.5	3.5	3.5	3.5	Y
-3	3	3	3	1	N
-1	1	1	1	Und	N
0	0	0	0	0	Y
1	-1	2	DNE	-1	N
4	3	3	3	Und	N
5	2	-3	DNE	2	N
9	-3	3	DNE	-1	N

7. Polynomials: (e) (multiply out) and (g). Rational (a) (use a common denominator) and (d) (multiply out).

$$\begin{aligned}
 8. \text{ a) } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{-2(x+h)^2 + 10(x+h) + 100 - (-2x^2 + 10x + 100)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2(x^2 + 2xh + h^2) + 10x + 10h + 100 + 2x^2 - 10x - 100}{h} = \lim_{h \rightarrow 0} \frac{-2x^2 - 4xh - 2h^2 + 10h + 2x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-4xh - 2h^2 + 10h}{h} = \lim_{h \rightarrow 0} -4x - 2h + 10 \stackrel{\text{Poly}}{=} -4x + 10.
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{3x - 3(x+h)}{(x+h)x}}{h} = \lim_{h \rightarrow 0} \frac{3x - 3x - 3h}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-3h}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{-3}{(x+h)x} \stackrel{\text{Rat}'1}{=} \frac{-3}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1\sqrt{x} - 1\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \\
 &= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h(\sqrt{x+h}\sqrt{x})(\sqrt{x} + \sqrt{x+h})} = \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{x+h}\sqrt{x})(\sqrt{x} + \sqrt{x+h})} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(\sqrt{x+h}\sqrt{x})(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{(x)(2\sqrt{x})} = \frac{-1}{2x^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad & \text{a) } \lim_{x \rightarrow 1.4^-} \lfloor x \rfloor = 1 & \text{b) } \lim_{x \rightarrow 1.4^+} \lfloor x \rfloor = 1 & \text{c) } \lim_{x \rightarrow 1.4} \lfloor x \rfloor = 1 \\
 & \text{d) } \lim_{x \rightarrow 2^-} \lfloor x \rfloor = 1 & \text{e) } \lim_{x \rightarrow 2^+} \lfloor x \rfloor = 2 & \text{f) } \lim_{x \rightarrow 2} \lfloor x \rfloor \text{ DNE} \\
 & \text{g) } \lim_{x \rightarrow 0} \lfloor x \rfloor \text{ DNE} & \text{h) } \lim_{x \rightarrow -1.5} \lfloor x \rfloor = -2
 \end{aligned}$$

