Math 130: Lab 3

5-Minute Quiz on Friday. It will include some of the definitions below.

1. **a)** Carefully state the definition of $f(x)$ being *continuous* at $x = a$.

b) State the definition of f having a *vertical asymptote* at $x = a$. Your definition should involve particular limits.

c) Carefully state the definition of f having a *removable discontinuity* at $x = a$.

2. Evaluate these limits, if they "exist." Use $+\infty$ or $-\infty$ as appropriate. Show work.

a) $\lim_{x\to 0^-} \frac{x-1}{x^2(x+8)}$ **b)** $\lim_{x\to -8^-} \frac{x-1}{x^2(x+8)}$ **c)** $\lim_{x\to 2^-} \frac{x^2-3x+2}{x-2}$ **d)** $\lim_{x\to 1^+} \frac{x-2}{1-\sqrt{x}}$

e) Explain briefly where the the function in part (c) is NOT continuous. Your answer should mention the type of function involved. No limits are required.

3. Carefully evaluate these limits at infinity. Indicate your work.

a) $\lim_{t\to+\infty} 18t^2 - 2t^3$

b) $\lim_{x\to-\infty} -4x^{12} - 21x^5 + 7$

c) $\lim_{x\to+\infty} \frac{2x}{4x^{3/2}-2}$

d) $\lim_{x\to-\infty} \frac{5x^2-x}{10x^2+1}$

e) $\lim_{x\to+\infty} \frac{2x^2+1}{x^{2/3}+4}$

f) $\lim_{x\to-\infty} \frac{5x^2-x}{10x^3+1}$

Determine these limits at infinity by focusing on highest powers. Use $\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

g) $\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+1}}$ **h)** $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+1}}$

i) Use the limits above to determine which functions have horizontal asymptotes and give their equations.

4. **Like a Quiz/Test Question.**

a) Carefully explain where $f(x) = \frac{x^2+5x+6}{x^2+2x-3}$ is NOT continuous. (See 2(e)).

b) Using limits determine where $f(x)$ has (1) vertical asymptotes, and (2) removable discontinuities. [Where should you look.] Use appropriate limits to justify each. See your definitions in #1.

c) *Check your understanding.* Give the equation of a rational function with a VA at $x = -2$ and a removable discontinuity at $x = 6$. (Hint: Look back at what happened in the first parts of this problem to create RD's and VA's.)

5. Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & \text{if } x > -1, \\ 1 & \text{if } x = -1, \\ \frac{x^2+x+2}{x+1} & \text{if} x < -1. \end{cases}$

a) Bill says, "Since $f(-1) = 1$, then $\lim_{x\to-1} f(x) = -1$. We can just plug in." Explain why Bill is wrong.

b) Is f continuous at $x = -1$? Justify your answer using limit calculations.

c) Is f continuous at $x = 2$? Justify your answer by using a reason, not a limit calculation.

d) Does f have a VA at -1? Justify your answer using the VA definition and limits.

e) Does f have an RD at -1? Justify your answer.

6. This is a good problem to see if you understand the concepts we have been studying. Fill in the table using the information given. For some, several correct answers are possible.

a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Left Cont	Right Cont	Cont? Removable? VA?
-1	0	0		DNE			
1		1		4	Yes		
2			2		No		
3		3					No, Removable

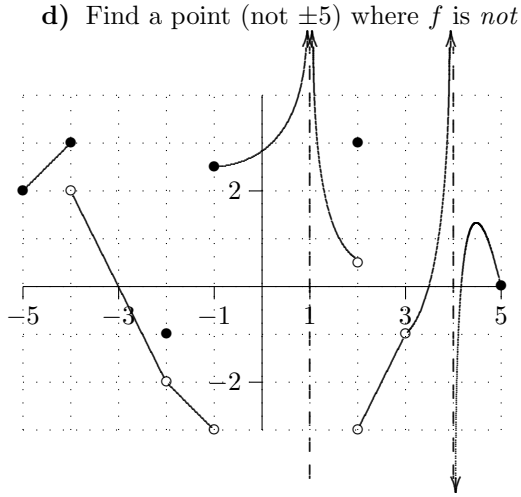
a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Left Cont	Right Cont	Cont? Removable? VA?
4		1				Yes	VA
5			$-\infty$				
6		1					Cont

7. **a)** Fill in the table. Use $+\infty$ and $-\infty$ where appropriate.

b) Find a point $x = a$ where f is *not* continuous even though both $f(a)$ and the limit exist.

c) Find a point $x = a$ where f is *not* continuous because, though the limit exists, $f(a)$ does not.

d) Find a point (not ± 5) where f is *not* continuous because, though $f(a)$ exists, the limit doesn't.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Cont, VA, RD
-4					
-3					
-2					
-1					
0					
1					
2					
3					
4					

8. **Stop and Think.** Think about the definitions of the relevant terms.

a) If $f(a) = 5$ and f is continuous at a, then what can you say about $\lim_{x\to a} f(x)$?

b) True or False: If f has a VA at a, then $\lim_{x\to a} f(x)$ DNE in the usual (finite) sense.

c) True or false: Even if $f(x)$ has a removable discontinuity at $x = a$ it still might have a VA at $x = a$.

d) True or False: If $f(a)$ DNE, then f MUST have an RD at a.

e) On a test Bill is trying to evaluate $\lim_{x\to 1} \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials. He calculates that $p(1) = 0$ and $q(1) = 0$ so he writes $\lim_{x\to 1} \frac{p(x)}{q(x)} = \frac{\emptyset}{\emptyset} = 1$. How much credit should Bill receive? Explain.

9. Evaluate $\lim_{x\to 1^+} \frac{\frac{1}{x-1} - \frac{1}{3}}{x-4}$. Use $+\infty$ or $-\infty$ if appropriate. Hint: Simplify.

10. **a)** Let $f(x) = \frac{\frac{1}{3} - \frac{1}{2x+1}}{x-1}$. Determine where the function is NOT continuous.

b) Determine where f has VAs and RDs. Use appropriate limits.

Some Brief Answers Your explanations should be much longer! Complete answers on line.

1. Check with me or a TA. Or see Wednesday's Handout. Get these right.

2. $-\infty$, ∞, 1, $+\infty$. (c): Not continuous at $x = 2$.

3. $-\infty$, $-\infty$, 0, $\frac{1}{2}$, $+\infty$, 0, $\frac{3}{2}$, $-\frac{3}{2}$.

4. (a) $x = -3, 1$. (b) VA at $x = 1$, RD at $x = -3$. (c) E.g., $f(x) = \frac{x-6}{(x+2)(x-6)}$.

5. (b) No. (c) Continuous at $x = 2$ (rational there) or use $\lim_{x\to 2} f(x) \overset{\text{Rat'l}}{=} \frac{2^2+3(2)+2}{2+1} = 4 = f(2)$. (d) Yes. (e) No.

7. (b) -2. (c) -3. (d) $-4, -1, -2$.

8. (a) $f(a) = 5$. (b) True. (c) False. (d) False. (e) None.

9. $-\infty$.

10. All $x \neq 1, -\frac{1}{2}$. RD: $x = 1$. VA: $x = -\frac{1}{2}$.

Math 130: Answers to Lab 3

1. **a)** f is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$ OR (1) if $f(a)$ exists, (2) $\lim_{x\to a} f(x)$ exists, and (3) $\lim_{x\to a} f(x) = f(a)$.

b) f has a VA at $x = a$ if EITHER $\lim_{x\to a^-} f(x) = \pm\infty$ OR $\lim_{x\to a^+} f(x) = \pm\infty$.

c) f has a *removable discontinuity* at $x = a$ if (1) $\lim_{x\to a} f(x)$ exists but (2) $\lim_{x\to a} f(x) \neq f(a)$.

2. Solutions

a) $\displaystyle\lim_{x\to 0^-} \frac{x-1^{\nearrow -1}}{x^2(x+8)_{\searrow 0^+\cdot 8=0^+}} = -\infty$

b) $\displaystyle\lim_{x\to -8^-} \frac{x-1^{\nearrow -9}}{x^2(x+8)_{\searrow 64\cdot 0^-=0^-}} = +\infty$

c) $\displaystyle= \lim_{x\to 2^-} \frac{(x-1)(x-2)}{x-2} = \lim_{x\to 2^-}(x-1) = 1$

d) $\displaystyle\lim_{x\to 1^+} \frac{x-2^{\nearrow -1}}{1-\sqrt{x}_{\searrow 0^-}} = +\infty$

e) The function in (c) is rational so is continuous at every point in its domain. So not continuous at $x = 2$.

3. Carefully and quickly evaluate these polynomial and rational function limits at infinity by using dominant powers.

a) $\displaystyle\lim_{t\to+\infty} 18t^2 - 2t^3 \overset{\text{HP}}{=} \lim_{t\to+\infty} -2t^3 = -\infty$

b) $\displaystyle\lim_{x\to-\infty} -4x^{12} - 21x^5 + 7 \overset{\text{HP}}{=} \lim_{x\to-\infty} -4x^{12} = -\infty$

c) $\displaystyle\lim_{x\to+\infty} \frac{2x}{4x^{3/2}-2} \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{2x}{4x^{3/2}} = \lim_{x\to+\infty} \frac{2}{4x^{1/2}} = 0$ HA : $y = 0$

d) $\displaystyle\lim_{x\to-\infty} \frac{5x^2-x}{10x^2+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{5x^2}{10x^2} = \frac{1}{2}$ HA : $y = \frac{1}{2}$

e) $\displaystyle\lim_{x\to+\infty} \frac{2x^2+1}{x^{2/3}+4} \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{2x^2}{x^{2/3}} = \lim_{x\to+\infty} \frac{2x^{4/3}}{1} = +\infty$ No HA

f) $\displaystyle\lim_{x\to-\infty} \frac{5x^2-x}{10x^3+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{5x^2}{10x^3} = \lim_{x\to-\infty} \frac{1}{2x} = 0$ HA : $y = 0$

g) $\displaystyle\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{3x}{\sqrt{4x^2}} = \lim_{x\to\infty} \frac{3x}{|2x|} \overset{x\geq 0}{=} \lim_{x\to\infty} \frac{3x}{2x} = \frac{3}{2}$ HA : $y = \frac{3}{2}$

h) $\displaystyle\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2}} \lim_{x\to-\infty} \frac{3x}{|2x|} \overset{x\leq 0}{=} \lim_{x\to-\infty} \frac{3x}{-2x} = -\frac{3}{2}$ HA : $y = -\frac{3}{2}$

4. $f(x) = \frac{x^2+5x+6}{x^2+2x-3} = \frac{(x+2)(x+3)}{(x+3)(x-1)}$. f is rational; it is continuous except at $x = -3$ and 1 where it is undefined.

a) At $x = -3$: $\displaystyle\lim_{x\to-3} f(x) = \lim_{x\to-3} \frac{(x+2)(x+3)}{(x+3)(x-1)} = \lim_{x\to-3} \frac{x+2}{x-1} = \frac{-1}{-4} = \frac{1}{4}$. So $x = -3$ is an RD since $\lim_{x\to-3} f(x)$ exists but does not equal $f(-3)$ which DNE.

b) However $x = 1$ is a VA because checking the one-sided limits:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(x+2)(x+3)}{(x+3)(x-1)} = \lim_{x\to 1^+} \frac{x+2^{\nearrow 3}}{x-1_{\searrow 0^+}} = +\infty \quad \text{OR} \quad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+2^{\nearrow 3}}{x-1_{\searrow 0^-}} = -\infty.$$

In either case, f has a VA at $x = 1$ since a one-sided limit is infinite there.

c) Use a rational function such as $f(x) = \frac{x-6}{(x+2)(x-6)}$.

5. Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & \text{if } x > -1, \\ 1 & \text{if } x = -1, \\ \frac{x^2+x+2}{x+1} & \text{if} x < -1. \end{cases}$

a) The limit process requires that we *approach* -1 from both sides. The actual value of f at -1 may or may not be the limit. This is not a rational function at -1.

b) Continuous at -1? Use the definition: (1) Check that the function exists: $f(-1) = 1$. (2) Check that the limit exists. We must use 1-sided limits to determine $\lim_{x\to 1} f(x)$ since the function is split at $x = -1$. $\lim_{x\to -1^-} f(x) =$

$$\lim_{x\to -1^-} \frac{\overbrace{x^2+x+2}^{2}}{\underbrace{x+1}_{0^-}} = -\infty.$$ (Already you can say that f is not continuous since now we know that $\lim_{x\to 1} f(x)$ DNE

in the usual sense.) $\lim_{x\to 1^+} f(x) = \lim_{x\to -1^+} \frac{x^2+3x+2}{x+1} = \lim_{x\to -1^+} \frac{(x+2)(x+1)}{x+1} = \lim_{x\to -1^+} x+2 \overset{\text{Poly}}{=} 1$. Since the left and right limits are different, $\lim_{x\to -1} f(x)$ DNE, so f is not continuous at $x=-1$.

c) Since f is rational and defined when $x \neq -1$, it is continuous for all $x \neq -1$, including $x = 2$. Or use $\lim_{x\to 2} f(x) \overset{x>-1}{=} \lim_{x\to 2} \frac{x^2+3x+2}{x+1} \overset{\text{Rat'l}}{=} \frac{2^2+3(2)+2}{2+1} = 4 = f(2)$.

d) f has a VA at $x=-1$ because $\lim_{x\to 1^-} f(x) = -\infty$.

e) f does not have an RD at $x=-1$ because $\lim_{x\to 1} f(x)$ DNE.

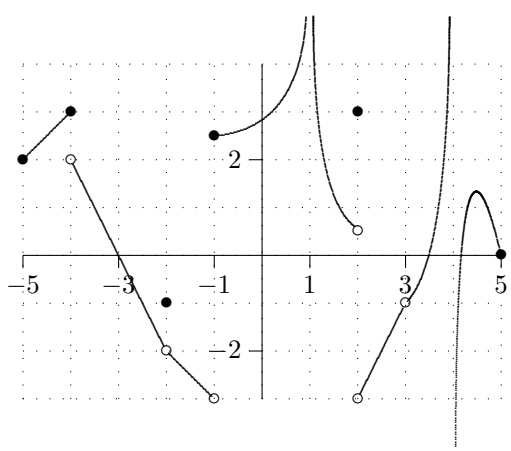
6.

a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Left Cont	Right Cont	Cont? Removable? VA?
-1	0	0	0	DNE	No	No	No, Removable, No VA
1	4	1	DNE	4	Yes	No	No, Not Rem, No VA
2	2	2	2	Not 2	No	No	No, Removable, No VA
3	3	3	3	Not 3	No	No	No, Removable, No VA
4	∞ or $-\infty$	1	DNE	1	No	Yes	No, Not Rem, VA
5	$-\infty$	$-\infty$	$-\infty$	Not $\pm\infty$	No	No	No, Not Rem, VA
6	1	1	1	1	Yes	Yes	Cont, Not Rem, No VA

7. **a)** f is *not* continuous even though both $f(a)$ and the limit exist: $a=-2$ only.

b) f is *not* continuous because, though the limit exists, $f(a)$ does not: $a=3$ only.

c) f is *not* continuous because, though $f(a)$ exists, the limit doesn't: $a=-4,-1,2$.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Cont, VA, RD
-4	3	2	DNE	3	None
-3	0	0	0	0	Cont
-2	-2	-2	-2	-1	RD
-1	-3	2.5	DNE	2.5	None
0	2.8	2.8	2.8	2.8	Cont
1	$+\infty$	$+\infty$	$+\infty$	DNE	VA
2	0.5	-3	DNE	3	None
3	-1	-1	-1	DNE	RD
4	$+\infty$	$-\infty$	DNE	DNE	VA

8. **a)** If f is continuous at a, then by definition $\lim_{x\to a} f(x) = f(a) = 5$.

b) If f has a VA at a, then $\lim_{x\to a} f(x)$ DNE in the usual sense. True, either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

c) False. If f has an RD at $x=a$, then $\lim_{x\to a} f(x)$ exists and is finite, so both one-sided limits exist and are finite. But if f had a VA at $x=a$, then at least one one-sided limit there would be infinite.

d) False: To have an RD at a we need $\lim_{x\to a} f(x)$ to exist (even if $f(a)$ DNE).

e) None. $\frac{0}{0}$ is an indeterminate form and requires more work before the limit can be evaluated.

9. $$\lim_{x\to 1^+} \frac{\frac{1}{x-1}-\frac{1}{3}}{x-4} = \lim_{x\to 1^+} \frac{\frac{3-(x-1)}{3(x-1)}}{x-4} = \lim_{x\to 1^+} \frac{3-(x-1)}{3(x-1)(x-4)} = \lim_{x\to 1^+} \frac{\overbrace{4-x}^{=-1(x-4)}}{3(x-1)(x-4)} = \lim_{x\to 1^+} \frac{-1}{\underbrace{3(x-1)}_{3\cdot 0^+=0^+}} = -\infty.$$

10. Simplify: $f(x) = \dfrac{\frac{1}{3} - \frac{1}{2x+1}}{x-1} = \dfrac{2x+1-3}{3(2x+1)(x-1)} = \dfrac{2x-2}{3(2x+1)(x-1)}$.

a) f is rational and is therefore continuous at all points in its domain: $x \neq 1, -\frac{1}{2}$.

b) VAs and RDs. Look at the limits at $x = 1, -\frac{1}{2}$. At $x = 1$:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2x-2}{3(2x+1)(x-1)} = \lim_{x\to 1} \frac{2}{3(2x+1)} \overset{\text{Rat'l}}{=} \frac{2}{9}$$

Since $\lim_{x\to 1} f(x)$ exists and $f(1)$ is undefined, there is an RD at $x = 1$. At $x = -1/2$:

$$\lim_{x\to -\frac{1}{2}^-} f(x) = \lim_{x\to -\frac{1}{2}^-} \frac{2x-2}{3(2x+1)(x-1)} = \lim_{x\to -\frac{1}{2}^-} \frac{2}{\underbrace{3(2x+1)}_{3\cdot 0^- = 0^-}} = -\infty$$

By definition, f has a VA at $x = -1/2$.