Math 130: Lab 4

1. Carefully and quickly evaluate these polynomial and rational function limits at infinity by using highest powers. Note any Horizontal Asymptotes.

a) $\displaystyle\lim_{x\to+\infty} \frac{2x^2+1}{x^{2/3}+4}$

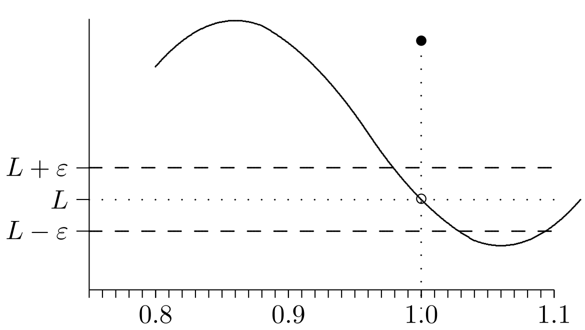
b) $\displaystyle\lim_{x\to-\infty} \frac{5x^2-x}{10x^3+1}$

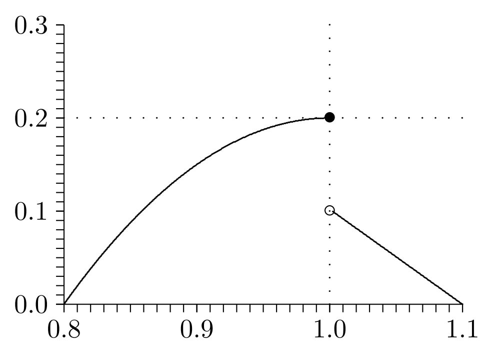
c) $\displaystyle\lim_{x\to\infty} \frac{2-3x}{\sqrt{4x^2+1}}$

d) $\displaystyle\lim_{x\to-\infty} \frac{2-3x}{\sqrt{4x^2+1}}$

e) Determine the horizontal asymptotes of $f(x) = \dfrac{|x|}{2x+1}$. Show your work. Be sure to look in both directions.

2. a) In the figure below (left), for the given choice of ε, find and draw a δ interval about $a = 1$ which satisfies the limit definition Note scale!





b) In the function on the right, $f(1) = .2$. However, show that $\lim_{x\to1} f(x) \neq 0.2$ by finding an $\varepsilon > 0$ (draw the horizontal band) for which no corresponding δ can be found. Explain why your ε works.

c) Is either of the functions above continuous at $a = 1$? Why?

d) Does either have a removable discontinuity at $a = 1$? Why?

3. Use the formal definition of limit to show that $\displaystyle\lim_{x\to2} 4x-3 = 5$.

4. In one of the questions below, the Intermediate Value Theorem can be used to guarantee that the equation has a solution. In the other it cannot. Determine which is which and carefully explain why a solution must exist in one case but not the other.

a) Can the Intermediate Value Theorem be used to demonstrate that $f(x) = 2x + \cos x = 0$ at some point in the interval $[0, \pi]$? Carefully explain why or why not.

b) Can the Intermediate Value Theorem be used to guarantee that $f(x) = 6x^4 + 4x^3 - 2x^2 - x = 3$ in the interval $[-1, 1]$? Carefully explain why or why not.

5. Use the fact that the trig functions, log functions, and exponential functions are continuous on their domains to determine the following limits. For (a), use a trig id for $\sin^2 x$. **Mathematical grammar:** Use equal signs and the limit symbol as needed throughout the calculation.

a) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x}$

b) $\displaystyle\lim_{x\to \pi/2} \frac{\sin x - 1}{\sqrt{\sin x} - 1}$

c) $\displaystyle\lim_{x\to 0^-} \frac{\cos x}{x}$

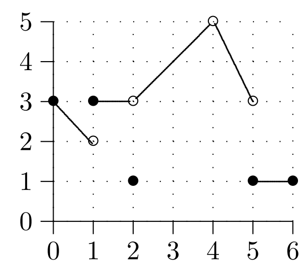
d) $\displaystyle\lim_{x\to 1^-} \frac{x}{\ln x}$

e) $\displaystyle\lim_{x\to 0^+} \frac{x+2}{e^x - 1}$

6. Which of these functions has a vertical asymptote as $x = -1$. Explain using the definition of VA.

a) $f(x) = \dfrac{x^2+1}{(x+1)^2}$

b) $h(x) = \dfrac{x^2+x}{x+1}$



7. a) Use the graph to determine the intervals of continuity the function.

8. a) Let $f(x) = \dfrac{\frac{1}{3} - \frac{1}{2x+1}}{x-1}$. Determine where the function is continuous. Express your answer as the union of two or more intervals. (Why don't you need to check endpoints in this case?)

b) Determine where f has VAs and RDs. Use appropriate limits.

c) True or False: f is right continuous at 1.

9. Let $f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & \text{if } x > -1, \\ 1 & \text{if } x = -1, \\ \frac{x^2+x+2}{x+1} & \text{otherwise.} \end{cases}$.

a) Is f left continuous at -1? Justify your answer with limits.

b) Right Continuous? Justify.

c) Continuous? Justify.

d) Determine where the function is continuous. Express your answer as the union of two or more intervals.

e) Does f have a VA at -1? Justify your answer using the VA definition and limits.

f) Does f have an RD at -1? Justify your answer.

10. XC: Use the formal definition of limit to show that $\displaystyle\lim_{x\to 3} |18 - 6x| = 0$.

Brief Answers. Complete answers online.

#1: $+\infty$, 0, $-\frac{3}{2}$, $\frac{3}{2}$, $\pm\frac{1}{2}$. #2: $0 < \delta \le 0.02$; $0 < \varepsilon < 0.1$. Neither is Cont, (a) has RD. #3: $\delta = \frac{\varepsilon}{4}$. #4: No, Yes. Lots of work needed! #5 $-\frac{1}{2}$, 2, $-\infty$, $-\infty$, $+\infty$. #6: Yes, No. Check Limits!! #7: $[0,1) \cup [1,2) \cup (2,4) \cup (4,5) \cup [5,6]$. #8: $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 1) \cup (1, \infty)$; VA: $x = -1/2$, RD: $x = 1$. #9: Only right continuous; $(-\infty, -1) \cup [-1, \infty)$. VA: $x = -1$.

Answers

1. Carefully and quickly evaluate these polynomial and rational function limits at infinity by using dominant powers.

a) $\displaystyle \lim_{x\to+\infty} \frac{2x^2+1}{x^{2/3}+4} \overset{\text{HP}}{=} \lim_{x\to+\infty} \frac{2x^2}{x^{2/3}} = \lim_{x\to+\infty} \frac{2x^{4/3}}{1} = +\infty$ No HA

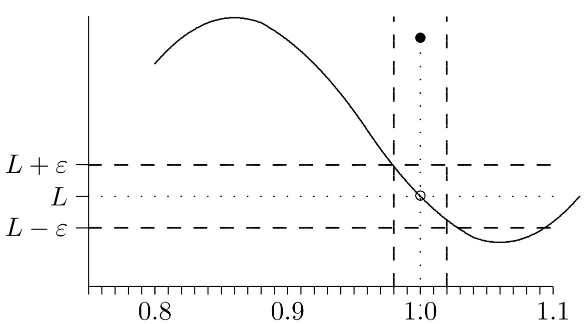
b) $\displaystyle \lim_{x\to-\infty} \frac{5x^2-x}{10x^3+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{5x^2}{10x^3} = \lim_{x\to-\infty} \frac{1}{2x} = 0$ HA : $y=0$

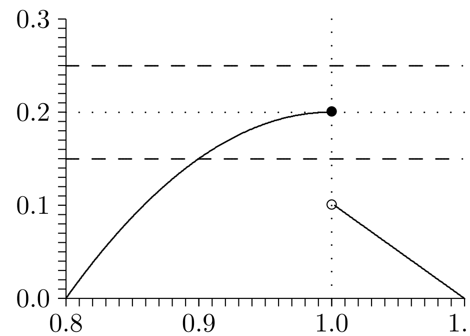
c) $\displaystyle \lim_{x\to\infty} \frac{2-3x}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{-3x}{\sqrt{4x^2}} = \lim_{x\to\infty} \frac{-3x}{|2x|} = \lim_{x\to\infty} \frac{-3x}{2x} = -\frac{3}{2}$ HA : $y = \frac{3}{2}$

d) $\displaystyle \lim_{x\to-\infty} \frac{2-3x}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{-3x}{\sqrt{4x^2}} = \lim_{x\to-\infty} \frac{-3x}{|2x|} = \lim_{x\to-\infty} \frac{-3x}{-2x} = \frac{3}{2}$ HA : $y = -\frac{3}{2}$

e) $\displaystyle \lim_{x\to\infty} \frac{|x|}{2x+1} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{|x|}{2x} \overset{\text{xpositive}}{=} \lim_{x\to\infty} \frac{x}{2x} = \frac{1}{2}$ $\displaystyle \lim_{x\to-\infty} \frac{|x|}{2x+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{|x|}{2x} \overset{\text{xnegative}}{=} \lim_{x\to-\infty} \frac{-x}{2x} = -\frac{1}{2}$ HA : $y = \pm\frac{1}{2}$

2. Note scale! (a) Any δ between 0 and .02 will work. (b) Choose ε with $0 < \varepsilon < 0.1$.





a) Neither is continuous. (a) is not continuous because $\lim\limits_{x\to1} f(x) \neq f(1)$ and (b) is not continuous because $\lim_{x\to1} f(x)$ DNE.

b) (a) has a removable discontinuity. $\lim\limits_{x\to1} f(x)$ exists but does not equal $f(1)$.

3. In this case $a=2$ and $L=5$.

- Scrap: Given $\varepsilon > 0$, find δ. Work backwards:

Translate from the general to this particular function.

$$|f(x)-L| < \varepsilon \overset{\text{Translate}}{\iff} |(4x-3)-5| < \varepsilon$$

Now simplify the absolute value.

$$\overset{\text{Simplify}}{\iff} |4x-8| < \varepsilon$$

Factor out the constant in front of x.

$$\overset{\text{Factor}}{\iff} 4|x-2| < \varepsilon$$

Solve for $|x-a|$.

$$\overset{\text{Solve}}{\iff} |x-2| < \frac{\varepsilon}{4}.$$

We now have $|x-a| < \delta$ where $a=2$ and $\delta = \frac{\varepsilon}{4}$.

$$\text{Choose } \delta = \frac{\varepsilon}{4}.$$

At this last step we have an inequality of the form $|x-a| < \delta$. We identify δ as $\frac{\varepsilon}{3}$. Now we are ready to write the actual proof.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$. If $0 < |x-2| < \delta = \frac{\varepsilon}{4}$, then

$$|f(x)-L| = |(4x-3)-5| \overset{\text{Simplify}}{=} |4x-8|$$
$$\overset{\text{Factor}}{=} 4|x-2|$$

Because we know $|x-2| < \frac{\varepsilon}{4}$, we can substitute and make an inequality

$$\overset{|x-2|<\frac{\varepsilon}{4}}{<} 4\cdot\frac{\varepsilon}{4}=\varepsilon.$$

4. IVT: The Intermediate Value Theorem. Assume that

- f is continuous on the closed interval $[a,b]$ and
- L is a number between $f(a)$ and $f(b)$.

Then there is at least one number c in (a,b) so that $f(c)=L$.

a) Can the Intermediate Value Theorem be used to demonstrate that $f(x)=2x+\cos x=0$ at some point in the interval $[0,\pi]$? Carefully explain why or why not. **Solution.** Check the hypotheses of the IVT.

- Is $f(x)$ continuous on the interval $[0,1]$? Yes! Because $f(x)=2x+\cos x$ is the sum of continuous functions (a polynomial $2x$ and a trig function $\cos x$), so it is continuous.
- Is $L=0$ between $f(a)=f(0)$ and $f(b)=f(\pi)$? Well, $f(0)=0+\cos 0=1$ and $f(\pi)=2\pi+\cos(\pi)=2\pi-1=5.283$. So $L=0$ is NOT between $f(0)$ and $f(\pi)$.
- So we canNOT apply the IVT to say that there is some number c in $(0,\pi)$ so that $f(c)=L=0$. The IVT does not apply

b) Can the Intermediate Value Theorem be used to guarantee that $f(x)=6x^4+4x^3-2x^2-x=3$ in the interval $[-1,1]$? Carefully explain why or why not. **Solution.** We need to check the two hypotheses of the IVT.

- Is $f(x)$ continuous on the interval $[-1,1]$? Yes! Because f is a polynomial.
- Is $L=3$ between $f(a)=f(-1)$ and $f(b)=f(1)$? Well $f(a)=f(-1)=6-4-2+1=1$ and $f(1)=6+4-2-1=7$. So, yes, $L=3$ is between $f(-1)=1$ and $f(1)=7$.
- So we can apply the IVT and say that there is some number c in $(-1,1)$ so that $f(c)=L=3$.

5. a) $\displaystyle\lim_{x\to0}\frac{\cos x-1}{\sin^2 x}\overset{\text{TrigID}}{=}\lim_{x\to0}\frac{\cos x-1}{1-\cos^2 x}=\lim_{x\to0}\frac{\cos x-1}{(1-\cos x)(1+\cos x)}=\lim_{x\to0}\frac{-1}{1+\cos x}\overset{\text{TrigCont.}}{=}-\frac{1}{2}.$

b) $\displaystyle\lim_{x\to\pi/2}\frac{\sin x-1}{\sqrt{\sin x}-1}=\lim_{x\to\pi/2}\frac{\sin x-1}{\sqrt{\sin x}-1}\cdot\frac{\sqrt{\sin x}+1}{\sqrt{\sin x}+1}=\lim_{x\to\pi/2}\frac{(\sin x-1)(\sqrt{\sin x}+1)}{\sin x-1}=\lim_{x\to\pi/2}\sqrt{\sin x}+1\overset{\text{TrigCont, Root}}{=}2$

c) $\displaystyle\lim_{x\to0^-}\frac{\overbrace{\cos x}^{1}}{\underbrace{x}_{0^-}}=-\infty.$

d) $\displaystyle\lim_{x\to1^-}\frac{\overbrace{x}^{1}}{\underbrace{\ln x}_{0^-}}=-\infty.$ ($\ln x$ is a little less than 0 as $x\to1^-$. Think about the graph of $\ln x$.)

e) $\displaystyle\lim_{x\to0^+}\frac{\overbrace{x+2^2}}{\underbrace{e^x-1}_{0^+}}=+\infty$ (e^x is a little bigger than 1 as $x\to0^+$. Think about the graph of e^x.)

6. f has a VA at -1 if either $\displaystyle\lim_{x\to-1^+}f(x)=\pm\infty$ or $\displaystyle\lim_{x\to-1^=}f(x)=\pm\infty$. So Check the one-sided limits.

a) $\displaystyle\lim_{x\to-1^+}\frac{\overbrace{x^2+1}^{2}}{\underbrace{(x+1)^2}_{(0^+)^2=0^+}}=+\infty$, so there is a VA at $x=-1$.

b) $\displaystyle\lim_{x\to-1^+}\frac{\overbrace{x^2+x}^{0}}{\underbrace{(x+1)}_{0}}=\lim_{x\to-1^+}\frac{x(x+1)}{x+1}=\lim_{x\to-1^+}x=-1.$ Check the limit from the other side, too! $\displaystyle\lim_{x\to-1^-}\lim_{x\to-1^-}\frac{x(x+1)}{x+1}=\lim_{x\to-1^-}x=-1.$ Neither limit is infinite, there is no VA at $x=-1$, even though the denominator is 0. (There is an RD.)

7. $[0,1)\cup[1,2)\cup(2,4)\cup(4,5)\cup[5,6]$

8. Simplify: $f(x)=\dfrac{\frac{1}{3}-\frac{1}{2x+1}}{x-1}=\dfrac{2x+1-3}{3(2x+1)(x-1)}=\dfrac{2x-2}{3(2x+1)(x-1)}$.

a) f is rational and is therefore continuous at all points in its domain: $x\neq 1,-\frac{1}{2}$. So the intervals of continuity are $(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},1)\cup(1,\infty)$. We don't have to check endpoints because f is not defined at either $x=1,-\frac{1}{2}$ so it cannot be left or right continuous at either point.

b) VAs and RDs. Look at the limits at $x=1,-\frac{1}{2}$. At $x=1$:

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{2x-2}{3(2x+1)(x-1)}=\lim_{x\to 1}\frac{2}{3(2x+1)}\overset{\text{Rat'l}}{=}\frac{2}{9}$$

Since $\lim\limits_{x\to 1} f(x)$ exists and $f(1)$ is undefined, there is an RD at $x=1$. At $x=-1/2$:

$$\lim_{x\to -\frac{1}{2}^-} f(x)=\lim_{x\to -\frac{1}{2}^-}\frac{2x-2}{3(2x+1)(x-1)}=\lim_{x\to -\frac{1}{2}^-}\frac{2}{\underbrace{3(2x+1)}_{3\cdot 0^-=0^-}}=-\infty$$

By definition, f has a VA at $x=-1/2$.

c) False: If f is not right continuous at 1 because $f(1)$ is not even defined.

9. Let $f(x)=\begin{cases}\frac{x^2+3x+2}{x+1} & \text{if } x>-1,\\ 1 & \text{if } x=-1,.\\ \frac{x^2+x+2}{x+1} & \text{otherwise.}\end{cases}$

a) Left continuous at -1? $f(-1)=1$ and $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\dfrac{\overbrace{x^2+x+2}^{2}}{\underbrace{x+1}_{0^-}}=-\infty$. So $\lim\limits_{x\to 1^-} f(x)\neq f(-1)$. Therefore, f is not left continuous at -1.

b) Right Continuous: $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to -1^+}\dfrac{x^2+3x+2}{x+1}=\lim\limits_{x\to -1^+}\dfrac{(x+2)(x+1)}{x+1}=\lim\limits_{x\to -1^+} x+2\overset{\text{Poly}}{=}1$. So $\lim\limits_{x\to 1^+} f(x)=f(-1)$. Therefore, f is right continuous at -1.

c) Continuous? No. $\lim\limits_{x\to 1^+} f(x)\neq\lim\limits_{x\to 1^-} f(x)$ so $\lim\limits_{x\to 1} f(x)$ DNE. Therefore f is not continuous at $x=-1$. OR f is not both left and right continuous at -1, so it cannot be continuous there.

d) Since f is rational and defined when $x\neq -1$, it is continuous for all $x\neq -1$. We showed that it was right continuous at -1, but not left continuous. So the intervals are $(-\infty,-1)\cup[-1,\infty)$.

e) f has a VA at -1 because $\lim\limits_{x\to 1^-} f(x)=-\infty$.

f) f does not have an RD at -1 because $\lim\limits_{x\to 1} f(x)$ DNE.

10. In this case $a=3$ and $L=0$.

- Scrap: Find δ. Assume that $\varepsilon>0$ is given. Then

$$\begin{aligned}||18-6x|-0|<\varepsilon &\overset{\text{Simplify}}{\iff} |18-6x|<\varepsilon\\ &\overset{\text{Factor}}{\iff} |-6||x-3|<\varepsilon \overset{\text{Solve}}{\iff} |x-3|<\frac{\varepsilon}{6}.\end{aligned}$$

  So $\delta=\frac{\varepsilon}{6}$. Do the proof.

- Proof: Given $\varepsilon>0$. Choose $\delta=\frac{\varepsilon}{6}$. If $0<|x-3|<\frac{\varepsilon}{6}$, then

$$||18-6x|-0|=|18-6x|=10|x+3|=|-6||x-3|<6\cdot\frac{\varepsilon}{6}=\varepsilon.$$