Math 130: Lab 5

1. a) Write out the $h \to 0$ limit definition of the derivative function $f'(x)$. This is on the next exam.

Use the definition of $f'(x)$ that you wrote in (a) to do the remaining parts.

b) If $f(x) = 2x^2 - 6x$ determine $f'(x)$.

c) If $g(x) = \frac{1}{x}$ determine $g'(x)$.

d) If $f(x) = \sqrt{x}$ determine $f'(x)$.

e) Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at $x = 4$. Use your work above.

f) If the position (in meters) of an object at time t seconds were $f(t) = \sqrt{t}$, what would its instantaneous velocity be at time $t = 10$ secs?

2. a) Using the calculations from class yesterday, the lab ticket, and lab today fill in the derivatives of each of these

$f(x)$	x^3	x^2	x	x^{-1}	x^{-2}	$\sqrt{x} = x^{1/2}$
$f'(x)$						

b) If you spot a pattern, if $f(x) = x^4$ what should $f'(x)$ be? If $f(x) = x^{-4}$, what should $f'(x)$ be?

3. a) Suppose that $f(x)$ is a differentiable function. Then we know that $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h.}$ Now assume that $g(x) = cf(x)$, where c is a constant. Use the definition of the derivative of $g(x)$, a limit property (which?), and the limit above to show to show that the derivative of $g'(x)$ is $cf'(x)$. I will start you out:

$$g'(x) \overset{\text{def of deriv}}{=} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \overset{\text{Substitute } cf(x) \text{ for } g(x)}{=}$$

b) You just proved that $g'(x) = [cf(x)]' = cf'(x)$. Use this rule YOU just proved and problem 1(d) to figure out the derivative of $g(x) = -10\sqrt{x}$ without ever using a limit.

c) Suppose $g(x) = 10x^2 - 30x$. Use the rule just proved and problem 1(b) to figure out the derivative of $g(x) = 10x^2 - 30x$ without ever using a limit. What's the c you need?

4. a) Suppose that $f(x)$ and $g(x)$ are both differentiable functions. Then we know that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Use a limit property (which?) and the two limits above to show that $[f(x) + g(x)]' = f'(x) + g'(x)$. I will start you out. Keep going and split into two pieces.

$$[f(x) + g(x)]' \overset{\text{def of deriv}}{=} \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (\qquad\qquad\qquad)}{h}$$

b) Use the rule just proved and problems 1(b and c) to find the derivative of $(2x^2 - 6x) + \frac{1}{x}$ without using a limit.

c) Combine your new derivative rules and use Problem 1 to find the derivative of $4\sqrt{x} + \frac{6}{x}$ without using a limit.

d) What should the rule be for $[f(x) - g(x)]'$. Do not prove it. Use this rule to determine the derivative of $\frac{1}{x} - \sqrt{x}$

5. a) Consider $\lim_{x\to 0} \frac{\overbrace{\sin x}^{0}}{\underbrace{x}_{0}}$. We need to do more work, but there is no simple algebra that will help us here. Fill in the table below to estimate the limit. Make sure your calculator is set to radians.

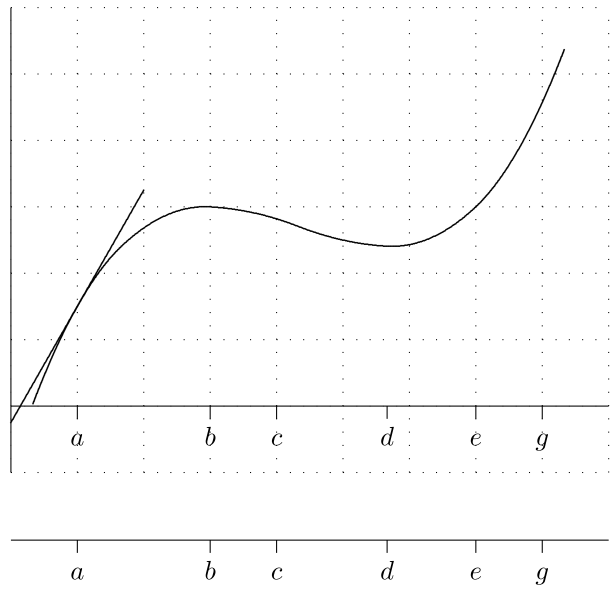
$x < 0$	$\frac{\sin x}{x}$	$x > 0$	$\frac{\sin x}{x}$
-0.1	0.99833417	0.1	
-0.01		0.01	
-0.001		0.001	
-0.0001		0.0001	

$$\lim_{x\to 0} \frac{\sin x}{x} = \underline{\qquad\qquad}$$

b) Determine $\lim_{x\to 0} \frac{1 - \cos x}{x}$. Hint: Multiply by the "conjugate" $\frac{1 + \cos x}{1 + \cos x}$ and then use a trig identity and then use your answer in part (a).

6. a) Consider the function f graphed below. Draw the tangents for each of the indicated points and use the slopes of the tangents to estimate f' and fill in the table below. I have drawn the first tangent for you.

x	a	b	c	d	e	g
$f'(x)$						



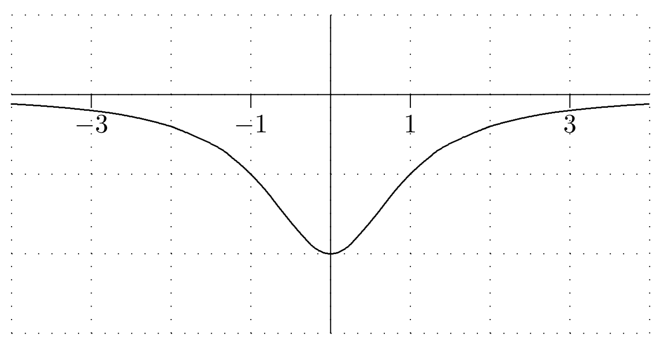
b) Put + signs along the number line below the graph at every point where the slope is positive, including points between the letters. Use − and 0 for the negative and zero slopes. **Sketch** the graph of f' on the same axes using the table of values you made in (a) and the number line.

c) For what intervals is $f'(x) < 0$? For what intervals is $f'(x) > 0$?

7. a) Consider the function f graphed below. For what values of x is $f'(x) = 0$?

b) Estimate the values of $f'(x)$ at each of the points in the table. Then use this information to graph $f'(x)$ on the same axes.

x	-3	-2	-1	0	1	2	3
$f'(x)$							



c) For what intervals is $f'(x) < 0$? For what intervals is $f'(x) > 0$?

8. Recall that $m_{\tan}$ is the tangent slope at a particular point a. This is the same as the derivative $f'(a)$ which is also the slope of the curve. Remember the formula for $m_{\tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. In other words, $m_{\tan} = f'(a)$.

a) Let $f(x) = \sin x$. Use the definition of $m_{\tan}$ at the point $a = 0$ to determine $f'(0)$. Use Problem 5(a).

b) Let $f(x) = \cos x$. Use the definition of $m_{\tan}$ at the point $a = 0$ to determine $f'(0)$. Use Problem 5(b).

Incomplete Answers. Complete Answers Online

1. (a) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if the limit exists. (b) $4x - 6$; (c) $-x^{-2}$; (d) $\frac{1}{2\sqrt{x}}$; (e) $y - 2 = \frac{1}{4}(x-4)$; (f) $\frac{1}{2\sqrt{10}}$.

2. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

3. (a) $cf'(x)$; $g'(x) = -\frac{5}{\sqrt{x}}$; (c) $g'(x) = 20x - 30$.

4. (a) $f'(x) + g'(x)$; (b) $4x - 6 - \frac{1}{x^2}$; (c) $\frac{2}{\sqrt{x}} - \frac{6}{x^2}$; (d) $f'(x) - g'(x)$, so $-\frac{1}{x^2} - \frac{1}{2\sqrt{x}}$.

5. $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

6.

x	a	b	c	d	e	g
$f'(x)$	1.8	0	−0.3	0	1	2.2

7. a) $x = 0$.

x	-3	-2	-1	0	1	2	3
$f'(x)$	−0.2	−0.3	−0.5	0	0.5	0.3	0.2

(c) $[-4, 0)$. (d) $(0, 4]$.

8. (a) 1; (b) 0.

Math 130: Lab 5 Answers

1. a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if the limit exists.

b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{2(x+h)^2 - 6(x+h) - [x^2-6x]}{h} = \lim_{h\to 0} \frac{2x^2+4xh+2h^2-6x-6h-2x^2+6x}{h}$$
$$= \lim_{h\to 0} \frac{4xh+2h^2-6h}{h} = \lim_{h\to 0} 4x+2h-6 = 4x-6.$$

c) $$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{1x-(x+h)}{(x+h)(x)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{(x+h)(x)h} = \lim_{h\to 0} \frac{-1}{(x+h)(x)} = -\frac{1}{x^2}.$$

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d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

e) We need a point and a slope. The point is $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$. From the previous part, the tangent slope is just $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. So the equation of the line is $y - 2 = \frac{1}{4}(x-4)$.

f) Instantaneous velocity is $f'(10)$. From part (d), $f'(t) = \frac{1}{2\sqrt{t}}$. So the instantaneous velocity is $f'(10) = \frac{1}{2\sqrt{10}}$ m/s.

2. a) Fill in the derivatives of each of these functions. Do you see a pattern?

$f(x)$	x^3	x^2	x	x^{-1}	x^{-2}	$x^{1/2}$
$f'(x)$	$3x^2$	$2x$	$1x^0 = 1$	$-x^{-2}$	$-2x^{-3}$	$\frac{1}{2}x^{-1/2}$

b) The pattern seems to be if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Lower the power by 1 and multiply by the old power. If $f(x) = x^4$, then $f'(x) = 4x^3$. If $f(x) = x^{-4}$, then $f'(x) = -4x^{-5}$

3. a) Start with the definition of the derivative of $g(x) = cf(x)$.

$$g'(x) \overset{\text{def of deriv}}{=} \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \overset{g(x)=cf(x)}{=} \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \overset{\text{constant}}{=} c \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \overset{f \text{ diff'ble}}{=} cf'(x)$$

b) Notice $g(x) = -10\sqrt{x} = -10f(x)$. So

$$g'(x) = -10f'(x) \overset{1\text{ (d)}}{=} -10 \cdot \frac{1}{2\sqrt{x}} = \frac{-10}{2\sqrt{x}} = -\frac{5}{\sqrt{x}}.$$

c) Notice $g(x) = 10x^2 - 30x = 5(2x^2-6x) = 5f(x)$. So the rule above says the derivative is:

$$g'(x) = 5f'(x) \overset{1\text{ (b)}}{=} 5(4x-6) = 20x - 30.$$

4. a) Start with the definition of the derivative of $f(x) + g(x)$.

$$[f(x)+g(x)]' \overset{\text{def of deriv}}{=} \lim_{h\to 0} \frac{f(x+h)+g(x+h)-[f(x)+g(x)]}{h} \overset{\text{re-order}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)+[g(x+h)-g(x)]}{h}$$
$$\overset{\text{sum}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \overset{f,g \text{ diff'ble}}{=} f'(x) + g'(x)$$

b) The rule says: $\left[(2x^2-6x) + \frac{3}{x}\right]' = \left[(2x^2-6x)\right]' + \left[\frac{3}{x}\right]' \overset{1\text{(b,c)}}{=} (4x-6) - \frac{1}{x^2}.$

c) Using Problem 2 and part (a)

$$\left[4\sqrt{x} + \frac{6}{x}\right]' \overset{\text{part (a)}}{=} \left[4\sqrt{x}\right]' + \left[\frac{6}{x}\right]' \overset{\#4}{=} 4\left[\sqrt{x}\right]' + 6\left[\frac{1}{x}\right]' \overset{\#3}{=} 4 \cdot \frac{1}{2\sqrt{x}} + 6 \cdot \frac{-1}{x^{-2}} = \frac{2}{\sqrt{x}} - \frac{6}{x^{-2}}.$$

d) The rule should be $[f(x) - g(x)]' = f'(x) - g'(x)$, so $\left[\frac{1}{x} - \sqrt{x}\right]' = -\frac{1}{x^2} - \frac{1}{2\sqrt{x}}.$

5. a)

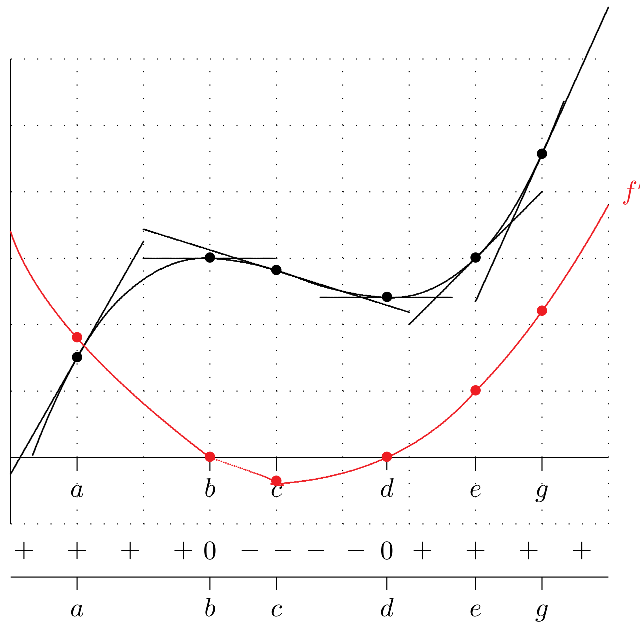
$x < 0$	$\frac{\sin x}{x}$	$x > 0$	$\frac{\sin x}{x}$
-0.1	0.99833417	0.1	0.99833417
-0.01	0.999833317	0.01	0.999833317
-0.001	0.9999998317	0.001	0.9999998317
-0.0001	0.999999998	0.0001	0.999999998

$$\lim_{x\to 0} \frac{\sin x}{x} = \underline{\quad 1 \quad}$$

b) $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} \overset{\text{Part (a), Contin}}{=} 1 \cdot \frac{0}{2} = 0.$

6. a) Use slopes of tangents to estimate f'.

x	a	b	c	d	e	g
$f'(x)$	1.8	0	-0.3	0	1	2.2



7. a) Consider the function f graphed below. For what values of x is $f'(x) = 0$? Only at $x = 0$.

b) Estimate the values of $f'(x)$ at each of the points in the table.

x	-3	-2	-1	0	1	2	3
$f'(x)$	-0.2	-0.3	-0.5	0	0.5	0.3	0.2

c) For what intervals is $f'(x) < 0$? $[-4, 0)$

d) For what intervals is $f'(x) > 0$? $(0, 4]$

8. a) $\displaystyle f'(0) = m_{\tan} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x} = \lim_{x\to 0} \frac{\sin x - 0}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$

b) $\displaystyle f'(0) = m_{\tan} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\cos x - \cos 0}{x} = \lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$