# Math 130: Lab 6

**1.** Assume that $f$ and $g$ are differentiable. Write out the derivative rules for the following expressions. Which one can't we do yet?

**a)** $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ **b)** $\frac{d}{dx}[f(x)-g(x)]$ **c)** $\frac{d}{dx}[f(x)g(x)]$ **d)** $\frac{d}{dx}[f(g(x))]$

**2. a)** State the derivative rule for: $\frac{d}{dx}\left[e^{kx}\right] =$ _______________. Use this rule to quickly determine the derivatives of the following functions.

**b)** $6e^{10x}$ **c)** $\frac{e^{-7x}}{2}$ **d)** $\frac{2e^{x/3}}{5}$ **e)** $e^{11}$

**f)** Jeopardy! Suppose I told you that $f'(x) = 8e^{2x}$. What was $f(x)$?

**3.** Use your derivative rules to determine the derivatives of the following functions.

**a)** $y = x + \frac{1-e^x}{2+e^x}$ **b)** $f(x) = \frac{1}{4x^2} + \frac{5}{\sqrt{x^5}} + \frac{9}{5x}$

**c)** $g(x) = x^2e^x$ **d)** $f(x) = \frac{5xe^{-3x}}{6}$ Not Quotient!

**e)** $y = -23\pi + \sqrt[7]{t^2}e^{-t}$ **f)** $f(t) = \frac{te^{2t}}{t^2+2}$ Two Rules!

**g)** $y = \frac{6e^{3x}}{x^{3/2}}$ **h)** $f(\theta) = \frac{3\theta^2-1}{3e^{2\theta}}$ **i)** $f(x) = (5x^2+x+1)(4x^3-x)$

**4. a)** Let $f(x) = e^{-2x} + 2x + 4$. Determine where $f(x)$ has a horizontal tangent. (Hint: What is $f'(x)$ when the tangent line is horizontal?)

**b)** Determine where the tangent slope is 2.

**c)** When we take the derivative of a function $f(x)$ we get a new function $f'(x)$. So we can take the derivative of $f'(x)$. This is called the second derivative and is denoted by $f''(x)$. Determine $f''(x)$ for the function in part (a). What would $f'''(x)$ be?

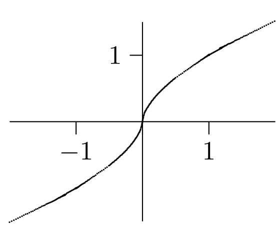
**5. a)** Like a test question. **Theory does not need to be hard.** Let $f(x)$ be a differentiable function. Use the **limit definition of the derivative with** $h \to 0$ to find the derivative of the function $g(x) = xf(x)$. Do not use the product rule. Start with:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \overset{\text{Use g(x)=xf(x)}}{=}$$

**b) Where did you use the fact that $f$ is continuous? How do you know that $f$ is continuous?**

**c)** Check your answer using the product rule for derivatives.

**6.** The graph of $f(x) = x^{3/5}$ is given below. Use the definition of the derivative at the point 0, namely $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$, to show that $f'(0)$ does not exist. Geometrically, why doesn't the derivative exist at 0? Is $f$ continuous at 0?



**7. a)** Using Calculus: The **position** of a ball thrown upward from 144 ft high rooftop is given by $s(t) = 144 + 128t - 16t^2$ ft, were where time $t$ is measured in seconds. Using derivative rules, find the velocity for the object. What are the units?

**b)** What is the acceleration of the ball? (**Acceleration** is the **instantaneous rate of change in the velocity**, so how can you very easily compute it? Think about problem 4(c).)

**c)** What is the velocity (not the position) at exactly $t = 2$ seconds?

**d)** When does the ball hit the ground? What is its velocity at this instant?

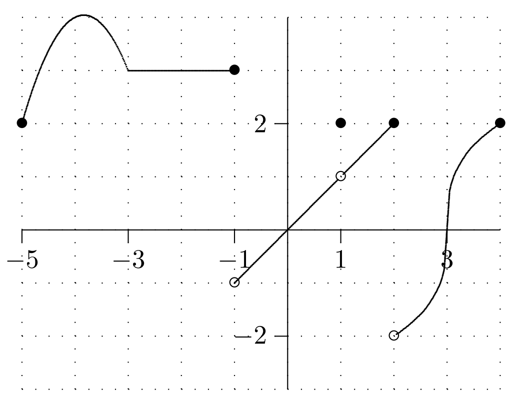
**e)** When is the ball at its maximum height? Explain how you know this.

**8. a)** Fill in the table.

**b)** Find all points where $f$ is *not continuous.*

**c)** Find all points $x = a$ where $f$ is *not differentiable* even though $f$ is continuous at $x = a$.

**d)** Find a point $x = a$ where $f$ is *not continuous* even though $f$ is differentiable at $x = a$.




| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ | Ctns | Diff'ble |
|---|---|---|---|---|---|---|
| $-4$ | | | | | | |
| $-3$ | | | | | | |
| $-2$ | | | | | | |
| $-1$ | | | | | | |
| $0$ | | | | | | |
| $1$ | | | | | | |
| $2$ | | | | | | |
| $3$ | | | | | | |

**9.** True or false. If false, draw a function which illustrates your answer. [Think about the previous problem.]

**a)** If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$.

**b)** If $f$ is continuous at $x = a$, then $f$ is differentiable at $x = a$.

**c)** If $f$ is not differentiable at $x = a$, then $f$ is not continuous at $x = a$.

**Brief Answers** Complete answers online.

**1.** We have not developed the derivative formula for the fourth function (composition).

**a)** $\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ **b)** $f'(x) - g'(x)$ **c)** $f'(x)\cdot g(x) + f(x)\cdot g'(x)$

**2.** **a)** $\dfrac{d}{dx}\left[e^{kx}\right] = ke^{kx}$ **b)** $60e^{10x}$ **c)** $\dfrac{-7}{2}e^{-7x}$ **d)** $\dfrac{2e^{x/3}}{15}$ **e)** $0$ **f)** $f(x) = 4e^{2x}$

**3.** **a)** $1 - \dfrac{3e^x}{(2+e^x)^2}$ **b)** $-\dfrac{1}{2}x^{-3} - \dfrac{25}{2}x^{-7/2} - \dfrac{9}{5}x^{-2}$ **c)** $2xe^x + x^2e^x = (2x + x^2)e^x$

**d)** Product rule, not quotient. $\dfrac{d}{dx}\left(\dfrac{5xe^{-3x}}{6}\right) \overset{\text{Prod}}{=} \dfrac{5}{6}[e^{-3x} + x(-3e^{-3x})] = \dfrac{5e^{-3x}(1-3x)}{6}$.

**e)** $\overset{\text{Prod}}{=} \dfrac{2}{7}t^{-5/7}e^{-t} - t^{2/7}e^{-t}$ **f)** $y' = \dfrac{(e^{2t} + 2te^{2t})(t^2+2) - (te^{2t})2t}{(t^2+2)^2} = \dfrac{(2t^3 - t^2 + 4t + 2)e^{2t}}{(t^2+2)^2}$

**g)** $y' \overset{\text{Quot}}{=} \dfrac{18e^{3x}x^{3/2} - 9x^{1/2}e^{3x}}{x^3} = \dfrac{18xe^{3x} - 9e^{3x}}{x^{5/2}}$ [Divide out $x^{1/2}$.]

**h)** $f'(\theta) \overset{\text{Quot}}{=} \dfrac{6\theta(3e^{2\theta}) - (3\theta^2 - 1)2\cdot 3e^{2\theta}}{(3e^{2\theta})^2} = \dfrac{-6\theta^2 + 6\theta + 2}{3e^{2\theta}}$.

**i)** $f'(x) \overset{\text{Prod}}{=} (10x+1)(4x^3 - x) + (5x^2 + x + 1)(12x^2 - 1) = 100x^4 + 16x^3 - 3x^2 - 2x - 1$.

**4.** (a) $f'(x) = -2e^{-2x} + 2 = 0$. $f'(x) = 0$ at $x = 0$. (b) Never. (c) $f''(x) = 4e^{-2x}$. $f'''(x) = -8e^{-2x}$.

**7. a)** $s'(t) = 128 - 32t$ ft/s. **(b)** $s''(t) = -32$ ft/s$^2$. **(c)** 64 ft/s. **(d)** $-160$ ft/s. **(e)** $t = 4$.

# Math 130: Lab 6 Answers

**1.** We have not developed the derivative formula for the fourth function (composition).

**a)** $\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ **b)** $f'(x) - g'(x)$ **c)** $f'(x) \cdot g(x) + f(x) \cdot g'(x)$

**2. a)** $\dfrac{d}{dx}\left[e^{kx}\right] = ke^{kx}$. Use this rule for the derivatives below.

**b)** $y' = 6 \cdot 10e^{10x} = 60e^{10x}$ **c)** $y' = \dfrac{1}{2}(-7e^{-7x}) = \dfrac{-7}{2}e^{-7x}$

**d)** $y' = \dfrac{2}{5} \cdot \dfrac{1}{3} = e^{x/3} = \dfrac{2e^{x/3}}{15}$ **e)** $0$ ($e^{11}$ is a constant)

**f)** $f(x) = 4e^{2x}$ or even better $f(x) = 4e^{2x} + c$ where $c$ can be any constant.

**3. a)** $y' \overset{\text{Quot}}{=} 1 + \dfrac{-e^x(2+e^x) - (1-e^x)(e^x)}{(2+e^x)^2} = 1 + \dfrac{-2e^x - e^{2x} - e^x + e^{2x}}{(2+e^x)^2} = 1 - \dfrac{3e^x}{(2+e^x)^2}$

**b)** $\frac{d}{dx}\left(\frac{1}{4}x^{-2} + 5x^{5/2} + \frac{9}{5}x^{-1}\right) = -\frac{1}{2}x^{-3} - \frac{25}{2}x^{-7/2} - \frac{9}{5}x^{-2}$.

**c)** $g'(x) \overset{\text{Prod}}{=} 2xe^x + x^2e^x = (2x + x^2)e^x$.

**d)** Product rule, not quotient. $\dfrac{d}{dx}\left(\dfrac{5xe^{-3x}}{6}\right) \overset{\text{Prod}}{=} \dfrac{5}{6}[e^{-3x} + x(-3e^{-3x})] = \dfrac{5e^{-3x}(1-3x)}{6}$.

**e)** $D_t(-23\pi + t^{2/7}e^{-t}) \overset{\text{Prod}}{=} \frac{2}{7}t^{-5/7}e^{-t} - t^{2/7}e^{-t}$.

**f)** $y' = \dfrac{(e^{2t} + 2te^{2t})(t^2+2) - (te^{2t})2t}{(t^2+2)^2} = \dfrac{(2t^3 - t^2 + 4t + 2)e^{2t}}{(t^2+2)^2}$.

**g)** $y' \overset{\text{Quot}}{=} \dfrac{18e^{3x}x^{3/2} - 9x^{1/2}e^{3x}}{x^3} = \dfrac{18xe^{3x} - 9e^{3x}}{x^{5/2}}$ [Divide out $x^{1/2}$.]

**h)** $f'(\theta) \overset{\text{Quot}}{=} \dfrac{6\theta(3e^{2\theta}) - (3\theta^2 - 1)2 \cdot 3e^{2\theta}}{(3e^{2\theta})^2} = \dfrac{3e^{2\theta}[6\theta - 2(3\theta^2 - 1)]}{(3e^{2\theta})^2} = \dfrac{-6\theta^2 + 6\theta + 2}{3e^{2\theta}}$.

**i)** $f'(x) \overset{\text{Prod}}{=} (10x+1)(4x^3-x)+(5x^2+x+1)(12x^2-1) = 40x^4-10x^2+4x^3-x+60x^4-5x^2+12x^3-x+12x^2-1 = 100x^4 + 16x^3 - 3x^2 - 2x - 1$.

**4. a)** Horizontal tangent means $f'(x) = 0$. So $f'(x) = -2e^{-2x} + 2 = 0$. So $-2e^{-2x} = -2$ or $e^{-2x} = 1$. Taking logs: $\ln(e^{-2x} = \ln 1$ or $-2x = 0$, so $x = 0$. (Another method: Remember that $e^0 = 1$. So $e^{-2x} = e^0$, comparing exponents, we must have $x = 0$.)

**b)** Tangent slope is 2 means $f'(x) = 2$. So $f'(x) = -2e^{-2x} + 2 = 2$ or $-2e^{-2x} = 0$ or $e^{-2x} = 0$. This is impossible. An exponential function is never 0.

**c)** $f''(x) = 4e^{-2x}$. $f'''(x) = -8e^{-2x}$.

**5.** Use the product rule to check: $D(xf(x)) = f(x) + xf'(x)$. For the proof:

$$\begin{aligned}\frac{d}{dx}(xf(x)) &= \lim_{h\to 0}\frac{(x+h)f(x+h) - xf(x)}{h} = \lim_{h\to 0}\frac{xf(x+h) + hf(x+h) - xf(x)}{h} \\ &= \lim_{h\to 0} x\left(\frac{f(x+h) - f(x)}{h}\right) + \frac{hf(x+h)}{h} = \lim_{h\to 0} x\left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0}\frac{hf(x+h)}{h} \\ &= xf'(x) + \lim_{h\to 0} f(x+h) = xf'(x) + f(x).\end{aligned}$$

We used continuity to say $\lim_{h\to 0} f(x+h) = f(x)$. ($f$ is continuous b/c it is differentiable.)

**6.** $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0}\dfrac{x^{3/5} - 0}{x} = \lim\limits_{x\to 0}\dfrac{1}{\underbrace{x^{2/5}}_{0^+}} = +\infty$. [Notice the denominator goes to $0^+$ because it is a square, i.e., $x^{2/5} = (x^{1/5})^2$.] Thus, the derivative does not exist. Geometrically, the tangents are becoming vertical as $x \to 0$. Yes, $f$ is continuous at 0 since $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^{3/5} = 0 = f(0)$.

**7.** **a)** Velocity is $v(t) = s'(t) = 128 - 32t$ ft/s.

**b)** The acceleration is the rate of change in the velocity, in other words the derivative of the velocity, so acceleration is $a(t) = v'(t) = -32$ ft/s$^2$.

**c)** $v(2) = s'(2) = 128 - 32(2) = 64$ ft/s.

**d)** When the height is 0, or $s(t) = 144 + 128t - 16t^2 = -16(t^2 - 8t - 9) = -16(t-9)(t+1) = 0$. So $t = 9$. $t = -1$ is impossible. The velocity is $v(9) = s'(9) = 128 - 32(9) = -160$ ft/s.

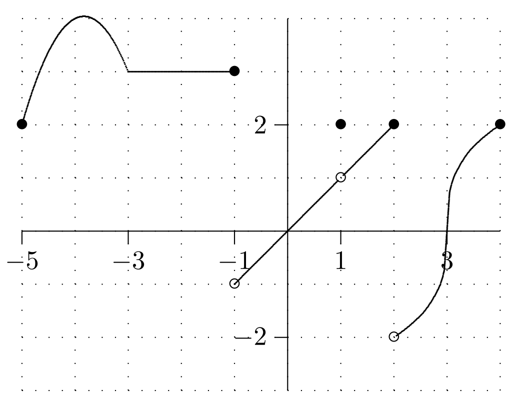
**e)** At the top of its flight, the velocity is 0. $v(t) = s'(t) = 128 - 32t = 0$ when $t = 4$.

**8.** **a)** See the table.

**b)** The points where $f$ is *not continuous*: $x = -1, 1, 2$.

**c)** $x = 3, -3$ are the points where $f$ is *not differentiable* even though $f$ is continuous.

**d)** Impossible. Differentiable implies continuous.




| $a$ | $\lim_{x\to a^-} f(x)$ | $\lim_{x\to a^+} f(x)$ | $\lim_{x\to a} f(x)$ | $f(a)$ | Ctns | Diff'ble |
|---|---|---|---|---|---|---|
| $-4$ | 4 | 4 | 4 | 4 | Yes | Yes |
| $-3$ | 3 | 3 | 3 | 3 | Yes | No |
| $-2$ | 3 | 3 | 3 | 3 | Yes | Yes |
| $-1$ | 3 | $-1$ | DNE | 3 | No | No |
| 0 | 0 | 0 | 0 | 0 | Yes | Yes |
| 1 | 1 | 1 | 1 | 2 | No | No |
| 2 | 2 | $-2$ | DNE | 2 | No | No |
| 3 | 0 | 0 | 0 | 0 | Yes | No |

**9.** (a) True (a theorem); (b) False: Use a function with a corner, like $y = |x|$ at $x = 0$ or see problem 8 at $x = -3$ or 3, or see problem 6 at $x = 0$; (c) False (same as b).