Math 130: Lab 7

You must absolutely positively be in control of the basic derivative rules by the time lab is finished today. The TAs and I will check your answers. Do these in your notebook or separate sheets of paper.

1. Assume that f and g are differentiable. Write out the derivative rules for the following expressions:

a) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ **b)** $\frac{d}{dx}[f(g(x))]$ **c)** $\frac{d}{dx}[f(x)g(x)]$

d) Which one is the product rule? The chain rule? The quotient rule?

2. If u is a function of x we have seen that we can write the chain rule for the derivative of e^u as $D_x(e^u) = e^u \frac{du}{dx}$. State the chain rule form of the derivative for

a) $D_x[\sin u]$ **b)** $D_x[\tan u]$ **c)** $D_x[\sec u]$ **d)** $D_x[u^n]$

3. OK, let's practice. Find the derivatives of the following functions using the product, quotient, and chain rules. For the first three identify which rule is appropriate. Then identify $f(x)$, $g(x)$, and when taking the derivative identify f, g, f' and g'. Here's an example: Find the derivative of $4\sqrt[5]{t^2}\tan t$.

$$\begin{array}{cccccccc} D_t[4t^{2/5} & \tan t] & = & \frac{8}{5}t^{-3/5} & \tan t & + & 4t^{2/5} & \sec^2 t \\ \uparrow & \uparrow & & \uparrow & \uparrow & & \uparrow & \uparrow \\ f(t) & g(t) & & f'(t) & g(t) & + & f(t) & g'(t) \end{array}$$

a) $x^2 \sec x$ **b)** $(4x^3 + 2x + 1)^3$ **c)** $\tan(9x^2)$ **d)** $x^2 \sin(x^5)$

e) $\frac{\cos(x^2)}{e^{2x}}$ **f)** $\sqrt{x^4 + x^2 + 5}$ **g)** $[x^3 e^{2x} + 1]^5$

h) Find the equation of the tangent line (not just the slope) in part **(f)** at the point $x = 2$.

4. Do each of the following by rewriting first to avoid the quotient rule.

a) $\frac{x^4}{2} - \frac{4}{x^2} + 2\sqrt{x^3}$ **b)** $\frac{3}{4\sqrt[3]{x^5}} + 19$ **c)** $\frac{4}{1+x^2}$ **d)** $\frac{(1+x^2)^3}{4}$ **e)** $\frac{\cos(2x)\tan(\pi x)}{5}$

5. Here are a few more to try. These are a bit trickier. Identify f and g only if you need to.

a) $((x^3+1)^2+2)^{-1/2}$ **b)** $\sin^5(\pi x)$ **c)** $\sec(x^3) + \sec^3(x)$ **d)** $\sin(\cos(\tan x))$

6. More notation. We use the symbol $D_x[f(x)]\Big|_{x=a}$ as another way of denoting $f'(a)$. Suppose that $f(x)$ and $g(x)$ are differentiable functions and that $f(4) = 3$, $g(0) = 2$, $f'(4) = 2$, $g'(0) = -1$, $f(2) = 1/2$, $g(3) = 5$, $f'(2) = 1$, and $g'(3) = -2$. Evaluate each of the following expressions. Hint: First find the general derivative then substitute in the required value $x = a$. You will need to use the product and chain rules. For example: to evaluate $D_x[g(f(x))]\Big|_{x=4}$, **first** use the chain rule on $g(f(x))$ and **then** substitute the given values.

$$D_x[g(f(x))]\Big|_{x=4} = g'(f(x))f'(x)\Big|_{x=4} = g'(f(4))f'(4) = g'(3)\cdot 2 = -2(2) = -4.$$

a) $D_x[x^2 f(x))\Big|_{x=4}$ **b)** $D_x[(f(x))^3]\Big|_{x=4}$ **c)** $D_x[f(g(x))]\Big|_{x=0}$ **d)** $D_x[f(x^2)]\Big|_{x=2}$

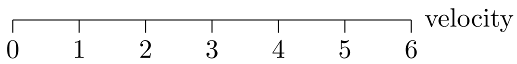
7. Applications. Remember: Derivatives are the same as tangent slopes are the same as instantaneous rates of change. Let $s(t) = \frac{t^3}{3} - 3t^2 + 5t$ be the displacement (meters) or position of an object at time t (seconds).

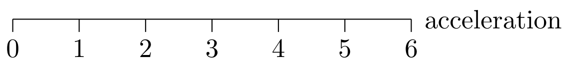
a) Determine the velocity, acceleration, and the jerk of the object. Acceleration is the derivative of velocity. The jerk is just the derivative of the acceleration.

b) When is the object stationary? Put a 0 above each such point on the number line below.

c) When is the acceleration 0? Mark it on the appropriate number line.

d) Put + or − sign above the number line for the velocity depending whether the velocity is positive or negative. Same for the acceleration.





e) Use your number line information to determine the time interval answers to the following: (1) When is the object moving backwards but accelerating? (2) When is the object stationary but accelerating? (3) When is the object moving forwards but decelerating? (4) When is the object is accelerating and moving forwards?

8. Applications. Let $s(t) = \sin t \cos t$ be the displacement (meters) or position of an object at time t (seconds).

a) Determine the velocity, acceleration, and the jerk of the object.

b) During the interval $[0, \pi]$, when is the object stationary?

9. Extra Credit: Though we know from the power rule that if $f(x) = x^{1/3}$ then $f'(x) = \frac{1}{3}x^{-2/3}$, I want you to try to *prove* this. Hint: use the fact that $[f(x)]^3 = (x^{1/3})^3 = x$. So start with the equation

$$[f(x)]^3 = x.$$

Take the derivatives of both sides of this equation (use the chain rule on the left) and then solve for $f'(x)$, substituting only at the very end that $f(x) = x^{1/3}$.

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3. Some Brief Answers

a) $x \sec x(2 + x \tan x)$ **b)** $3(4x^3 + 2x + 1)^2(12x^2 + 2)$ **c)** $18x \sec^2(9x^2)$

d) **e)** $\dfrac{-2[x\sin(x^2) + \cos(x^2)]}{e^{2x}}$ **f)** $\dfrac{2x^3 + x}{\sqrt{x^4 + x^2 + 5}}$ **g)** **h)** $y = \dfrac{18}{5}x - \dfrac{11}{5}$

4. **a)** $2x^3 + 8x^{-3} + 3x^{1/2}$ **b)** $-\dfrac{5}{4}x^{-8/3}$ **c)** **d)** $\frac{3}{2}x(x^2 + 1)^2$ **e)**

5. **a)** $-3x^2(x^3 + 1)[(x^3 + 1)^2 + 2]^{-3/2}$ **b)** **c)** $3x^2 \sec(x^3)\tan(x^3) + 3\sec^3(x)\tan(x)$
d) $-[\cos(\cos(\tan x))][\sin(\tan x)]\sec^2 x$

6. Take the derivative and substitute in the required values:

a) $2xf(x) + x^2 f'(x)\big|_{x=4} = 2(4)(3) + 4^2(2) = 56$ **b)** $3(f(x))^2 f'(x)\big|_{x=4} = 3(3)^2 2 = 54$
c) $f'(g(x))g'(x)\big|_{x=0} = (1)(-1) = -1$ **d)** $2xf'(x^2)\big|_{x=2} = 2(2)(2) = 8$

7. a) $v(t) = t^2 - 6t + 5$, $a(t) = 2t - 6$, and $j(t) = 2$. (b) At $t = 1$ and 5 seconds. (c) At $t = 3$ seconds. (e) (1) The object is moving backwards ($v < 0$) but accelerating ($a > 0$) when: $3 < t < 5$. (2) The object is stationary ($v = 0$) but accelerating: $t = 5$. (3) The object is moving forwards ($v > 0$) but decelerating ($a < 0$): $0 \le t < 1$. (4) The object is accelerating ($a > 0$) and moving forwards ($v > 0$): $5 < t \le 6$.

8. a) $v(t) = \cos^2 t - \sin^2 t,\ a(t) = -4\sin t \cos t,\ j(t) = 4\sin^2 t - 4\cos^2 t$
b) When $v(t) = \cos^2 t - \sin^2 t = 0$, or $\cos^2 t = \sin^2 t$. So $\cos t = \pm \sin t$. At $t = \frac{\pi}{4}$ and $\frac{3\pi}{4}$ seconds.

Math 130 Lab 7: Answers

1. a) $D_x\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

b) $D_x[f(g(x))] = f'(g(x))g'(x)$.

c) $D_x[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$.

2.

a) $D_x[\sin u] = \cos u \dfrac{du}{dx}$ **b)** $D_x[\tan u] = \sec^2 u \dfrac{du}{dx}$

c) $D_x[\sec u] = \sec u \tan u \dfrac{du}{dx}$ **d)** $D_x[u^n] = nu^{n-1}\dfrac{du}{dx}$

3. OK, using the basic derivative laws and a bit of simplification:

a) $D_x(fg) = \overbrace{2x\sec x}^{f'\cdot g} + \overbrace{x^2\sec x\tan x}^{f\cdot g'} = x\sec x(2 + x\tan x)$ **b)** $D_x(u^3) = \overbrace{3(4x^3+2x+1)^2}^{3u^2}\overbrace{(12x^2+2)}^{du/dx}$

c) $D_x[\tan u] = \overbrace{\sec^2(9x^2)}^{\sec^2 u}\overbrace{18x}^{du/dx} = 18x\sec^2(9x^2)$

d) $D_x[f(x)\sin(u)] = \overbrace{2x\sin(x^5)}^{f'\cdot\cos(u)} + \overbrace{x^2(\cos x^5)5x^4}^{f(x)\cdot\cos u\cdot du/dx} = x[2\sin(x^5) + 5x^5\cos(x^5)]$

e) $D_x\left(\dfrac{\cos(u)}{g(x)}\right) = \dfrac{-\sin(u)du/dx\cdot g(x) - \cos(u)g'(x)}{[g(x)]^2} = \dfrac{-\sin(x^2)(2x)e^{2x} - \cos(x^2)2e^{2x}}{(e^{2x})^2} = \dfrac{-2[x\sin(x^2) + \cos(x^2)]}{e^{2x}}$

f) $D_x(u^{1/2}) = \overbrace{\tfrac{1}{2}(x^4+x^2+5)^{-1/2}}^{\frac{1}{2}u^{-1/2}}\overbrace{(4x^3+2x)}^{du/dx} = (2x^3+x)(x^4+x^2+5)^{-1/2}$

g) $D_x(u^5) = \overbrace{5[x^3e^{2x}+1]^4}^{5u^4}\overbrace{(3x^2e^{2x}+2x^3e^{2x})}^{du/dx=f'g+fg'} = 5[x^3e^{2x}+1]^4(3+2x)x^2e^{2x}$

h) Slope: $m_{\tan} = f'(2) = (16+2)(16+4+5)^{-1/2} = \frac{18}{5}$. Point $(2, f(2)) = (2,5)$. Equation:

$$y - f(2) = m_{\tan}(x-2) \text{ or } y - 5 = \tfrac{18}{5}(x-2) \Rightarrow y = \tfrac{18}{5}x - \tfrac{11}{5}.$$

4. Rewrite as appropriate:

a) $D_x(\frac{1}{2}x^4 - 4x^{-2} + 2x^{3/2}) = 2x^3 + 8x^{-3} + 3x^{1/2}$ **b)** $D_x[\frac{3}{4}x^{-5/3} + 19] = -\frac{5}{4}x^{-8/3}$

c) $= D_x[4(1+x^2)^{-1}] = D_x[4(u)^{-1}] = -\overbrace{-4(1+x^2)^{-2}}^{-4u^{-2}}\overbrace{2x}^{du/dx} = -8x((1+x^2)^{-2}$

d) $D_x[\frac{1}{4}(x^2+1)^3] = D_x[\frac{1}{4}(u)^3] = \overbrace{\tfrac{3}{4}(x^2+1)^2}^{\frac{1}{4}3u^2}\overbrace{2x}^{du/dx} = \frac{3}{2}x(x^2+1)^2$

e) $D_x[\overbrace{\tfrac{1}{5}\cos(2x)}^{\frac{1}{5}\cos u}\overbrace{\tan(\pi x)}^{\tan v}] = \overbrace{-\tfrac{1}{5}\sin(2x)}^{-\frac{1}{5}\sin u}\overbrace{(2)}^{du/dx}\overbrace{\tan(\pi x)}^{v} + \overbrace{\tfrac{1}{5}\cos(2x)}^{\frac{1}{5}\cos u}\overbrace{\sec^2(\pi x)}^{\sec^2 v}\overbrace{(\pi)}^{dv/dx} = -\frac{2}{5}\sin(2x)\tan(\pi x) + \frac{\pi}{5}\cos(2x)\sec^2(\pi x)$

5. Various multiple uses of the chain rule. Watch out for the difference between $\sec^3(x)$ and $\sec(x^3)$.

a) $= D_x[\overbrace{(x^3+1)^2+2)^{-1/2}}^{u^{-1/2}}] = \overbrace{-\tfrac{1}{2}[(x^3+1)^2+2]^{-3/2}}^{-\frac{1}{2}u^{-3/2}}\overbrace{2(x^3+1)3x^2}^{du/dx} = -3x^2(x^3+1)[(x^3+1)^2+2]^{-3/2}$

b) $D_x[\overbrace{\sin^5(\pi x)}^{(\sin u)^5}] = \overbrace{5\sin^4(\pi x)}^{5(\sin u)^4}\overbrace{\cos(\pi x)}^{\cos u}\overbrace{(\pi)}^{du/dx} = 5\pi\sin^4(\pi x)\cos(\pi x)$

c) $D_{(}\overbrace{\sec(x^3)}^{\sec u} + \overbrace{[\sec(x)]^3}^{u^3} = \overbrace{\sec(x^3)\tan(x^3)}^{\sec u\tan u}\overbrace{3x^2}^{du/dx} + \overbrace{3\sec^2(x)}^{3u^2}\overbrace{\sec(x)\tan(x)}^{du/dx} = 3x^2\sec(x^3)\tan(x^3) + 3\sec^3(x)\tan(x)$

d) $D_x[\overbrace{\sin(\cos(\tan x))}^{\sin(\cos u)} = \overbrace{\cos(\cos(\tan x))}^{\cos(\cos u)}\overbrace{[-\sin(\tan x)]}^{-\sin u}\overbrace{\sec^2 x}^{du/dx} = -[\cos(\cos(\tan x))][\sin(\tan x)]\sec^2 x$

6. Take the derivative and substitute in the required values:

a) $2xf(x) + x^2 f'(x)\big|_{x=4} = 2(4)(3) + 4^2(2) = 56$

b) $3(f(x))^2 f'(x)\big|_{x=4} = 3(3)^2 2 = 54$

c) $f'(g(x))g'(x)\big|_{x=0} = (1)(-1) = -1$

d) $2xf'(x^2)\big|_{x=2} = 2(2)(2) = 8$

7. a) $v(t) = t^2 - 6t + 5$, $a(t) = 2t - 6$, and $j(t) = 2$.

b) When $v(t) = t^2 - 6t + 5 = (t-5)(t-1) = 0$. At $t = 1$ and 5 seconds.

c) When $a(t) = 2t - 6$. At $t = 3$ seconds.

d) The signs of the velocity and acceleration for the object:

velocity	$+$	0	$-$	$-$	$-$	0	$+$
t	0	1	2	3	4	5	6

acceleration	$-$	$-$	$-$	0	$+$	$+$	$+$
t	0	1	2	3	4	5	6

e) (1) The object is moving backwards ($v < 0$) but accelerating ($a > 0$) when: $3 < t < 5$. (2) The object is stationary ($v = 0$) but accelerating: $t = 5$. (3) The object is moving forwards ($v > 0$) but decelerating ($a < 0$): $0 \le t < 1$. (4) The object is accelerating ($a > 0$) and moving forwards ($v > 0$): $5 < t \le 6$.

8. a) $v(t) = \cos^2 t - \sin^2 t$, $a(t) = -4\sin t \cos t$, and $j(t) = 4\sin^2 t - 4\cos^2 t$.

b) When $v(t) = \cos^2 t - \sin^2 t = 0$, or $\cos^2 t = \sin^2 t$. So $\cos t = \pm \sin t$. At $t = \frac{\pi}{4}$ and $\frac{3\pi}{4}$ seconds.

9. Start with the chain rule, then use $f(x) = x^{1/3}$ at the third line:

$$\begin{aligned}
[f(x)]^3 = x \quad & \text{so} \quad 3[f(x)]^2 f'(x) = 1 \\
& \text{so} \quad f'(x) = \frac{1}{3[f(x)]^2} \\
\text{substitute } f(x) = x^{1/3} \quad & \text{so} \quad f'(x) = \frac{1}{3(x^{1/3})^2} \\
& \text{so} \quad f'(x) = \frac{1}{3x^{2/3}} \\
& \text{so} \quad f'(x) = \frac{1}{3}x^{-2/3}
\end{aligned}$$

Lab 7, Group Hand In. Names: ________________________________

1. These are similar to questions in problem 3. Determine

a) $D_x[2x^3\tan(x^4)]$

b) $D_x\left([e^{4x}\sin x + 2]^9\right)$

2. These are similar to questions in problem 4. Determine

a) $D_x\left(\dfrac{6}{2+\cos x}\right)$

b) $D_x\left(\dfrac{\sin(2x)\sec(\pi x)}{5}\right)$

3. Determine $D_x\left(\tan^2(\pi x)\right)$