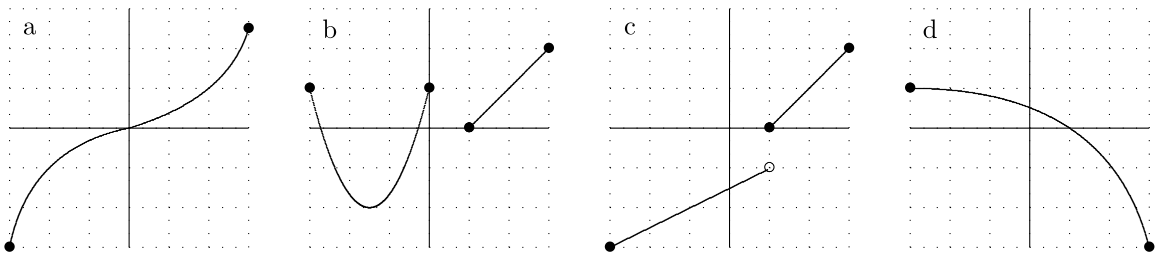
Math 130 Lab 8

1. Explain which of these functions have inverses and which do not. If a function has an inverse, graph it on the same set of axes.



2. Draw a graph that satisfies the given condition or explain why it is impossible to do so.

a) A function that is one-to-one and continuous.

b) A function that has an inverse but is not one-to-one.

c) A function that passes the vertical line test but not the horizontal line test.

3. Determine the following derivatives.

a) $D_x\left[\ln(e^{2x}+1)\right]$ **b)** $D_t\left[t^3\ln(t+1)\right]$ **c)** $D_x(8^x)$ **d)** $D_x(4^{x^2+1})$ **e)** $D_x(2^{x^3e^{4x}})$

4. a) Complete the basic log rules: $\ln(ab) =$ ______________, $\ln(\frac{a}{b}) =$ ______________, and $\ln(a^b) =$ ______________.
Use these rules to simplify these functions and then find the derivatives:

b) $f(x) = \ln x + \ln\frac{1}{x}$ **c)** $y = \ln\left(\dfrac{2+\cos(x^2)}{x^8+1}\right)$ **d)** $s(t) = \ln(t^t)$ **e)** $y = \ln(9^x)$

f) Find the equation of the tangent line in part (d) at the point $(1, 0)$.

5. Find the derivatives of the following functions. Use logarithmic differentiation where helpful.

a) $y = (\sin 9x)^x$ **b)** $y = x^{2\sin x}$ **c)** $y = x^{\ln x}$ **d)** $y = 6^x$

6. For each relation, find $\frac{dy}{dx}$ by implicit differentiation.

a) $x^2y + xy^2 = 2x$ **b)** $2x^3 - \sin(xy) = 2$

c) Find the tangent line to the curve in (a) at the point $(-1, 2)$.

7. Group Hand In. On Monday we were able to find the derivative of $f(x) = \ln x$ because $\ln x$ had an inverse, namely e^x, and we knew the derivative for the inverse. Here's the proof in step-by-step form: Fill in the blanks.

a) Let $y = \ln x$. We want to find $\frac{dy}{dx} = \frac{d}{dx}(\ln x)$. Start with:

$$y = \ln x$$

Apply the inverse of $\ln x$ to each side and then simplify.

______________ = _____

______________ = _____

b) Now take the derivative using implicit differentiation on the left:

$$\frac{d}{dx}[\quad] = \frac{d}{dx}[\quad]$$

______________ = _____

c) Solve for $\frac{dy}{dx}$. Then substitute $\ln x$ for y. Simplify at the very last step. $\frac{dy}{dx} =$ ______________. Did you get the correct formula for the derivative of $y = \ln x$?

8. Here are a few more derivatives to try:

a) $y = \dfrac{3x^2}{\ln(3x^2)}$ **b)** $y = \ln(\sin(e^{x^3+1}))$ **c)** $y = (\cos x)^{\tan(x^2)}$ **d)** $2^{x^2 \sin x}$

9. a) Complete the definition: $g(x)$ is the **inverse** of $f(x)$ if
(1) $f(g(x)) =$ ___________ for all x in the domain of g and
(2) $g(f(x)) =$ ___________ for all x in the domain of f.

b) Let $f(x) = x^3 + 7$. This function consists of two operations on x; what are they? In which order do they occur? If you wanted to 'undo' f what operation would you have to do first? Second? Use the two 'undoing' operations in the correct order to determine the inverse function $g(x)$. What is it?

c) Use the definition in part (a) to verify that the functions f and g in part (c) are in fact inverses.

d) Thinking problem: Draw a differentiable function that does NOT have an inverse. What can you say about the slope of such a function?

10. a) We say that $f(x)$ is **one-to-one** if whenever $x_1 \neq x_2$, then $f(x_1)$___________.

b) Let $f(x) = x^4 + 2x^2 + 3$. Show that f is NOT one-to-one by finding two different values x_1 and x_2 for which $f(x_1) = f(x_2)$.

11. What if we had done things differently? Suppose that we had first learned that $D_x[\ln x] = \frac{1}{x}$. Now let $f(x) = e^x$ and *assume you do not know its derivative.* Use the standard process for finding derivatives of inverses to determine the derivative of e^x. Here's the proof in step-by-step form: Fill in the blanks.

a) Let $y = e^x$. We want to find $\frac{dy}{dx}$. Start with:

$$y = e^x$$

Apply the inverse of e^x to each side and then simplify.

$$\underline{\hspace{3cm}} = \underline{\hspace{1cm}}$$

$$\underline{\hspace{3cm}} = \underline{\hspace{1cm}}$$

b) Now take the derivative using implicit differentiation on the left:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x]$$

$$\underline{\hspace{3cm}} = \underline{\hspace{1cm}}$$

c) Solve for $\frac{dy}{dx}$. Then substitute e^x for y. Simplify at the very last step. $\frac{dy}{dx} =$ ______________. Did you get the correct formula for the derivative of $f(x) = e^x$?

Selected answers

1. a, c, and d have inverses. **2.** (a) Examples: #1(a,d). (b) Impossible. (c) Example: #1(b).

3. **a)** $\dfrac{2e^{2x}}{e^{2x}+1}$ **b)** $3t^2\ln(t+1) + \dfrac{t^3}{t+1}$ **c)** $8^x \ln 8$ **d)** $4^{x^2+1}(\ln 4)2x$ **e)** $2^{x^3 e^{4x}} \ln 2[x^2 e^{4x}(3+4x)]$

4. a) Complete the basic log rules: $\ln(ab) = \ln a + \ln b$, $\ln(\frac{a}{b}) = \ln a - \ln b$, and $\ln(a^b) = b\ln a$.

b) 0 **c)** $-\dfrac{2x\sin(x^2)}{2+\cos(x^2)} - \dfrac{8x^7}{x^8+1}$ **d)** $\ln(t) + 1$ **e)** $\ln 9$ **f)** $y = x - 1$

5. **a)** $(\sin 9x)^x(\ln(\sin 9x) + 9x\cot 9x)$ **b)** $x^{2\sin x}\left(2\cos x \ln x + \dfrac{2\sin x}{x}\right)$ **c)** $x^{\ln x} \cdot \dfrac{2\ln x}{x}$ **d)** $y' = 6^x \ln 6$

6. **a)** $\dfrac{dy}{dx} = \dfrac{2 - 2xy - y^2}{x^2 + 2xy}$ **b)** $\dfrac{dy}{dx} = -\dfrac{y\cos(xy) - 6x^2}{x\cos(xy)}$ **c)** $y - 2 = -\dfrac{2}{3}(x+1)$

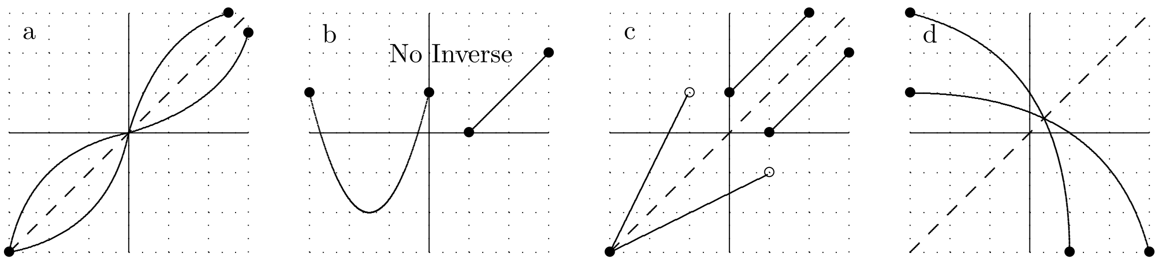
8. **a)** $\dfrac{6x[\ln(3x^2) - 1]}{[\ln(3x^2)]^2}$ **b)** $3x^2(e^{x^3+1})\cot(e^{x^3+1})$

c) $(\cos x)^{\tan(x^2)}\left[2x\sec^2(x^2)\cdot\ln(\cos x) - \tan(x^2)\tan x\right]$ **d)** $2^{x^2\sin x}(2x\sin x + x^2\cos x)\ln 2$

9. (b) $g(x) = (x-7)^{1/3}$. (c) $g(f(x)) = g(x^3+7) = ((x^3+7)-7)^{1/3} = (x^3)^{1/3} = x$ and $f(g(x)) = f((x-7)^{1/3}) = [(x-7)^{1/3}]^3 + 7 = (x-7)+7 = x$. Both check.

Lab 8 Answers

1. (b) does not have an inverse since it does not pass the HLT. The others do



2. a) See the graph in #1(a) or (d).

b) Impossible. If f has an inverse, it must pass the HLT (must be one-to-one).

c) See the graph in #1(b).

3. Determine the following derivatives

a) $D_x\left[\ln(e^{2x}+1)\right] = \dfrac{2e^{2x}}{e^{2x}+1}$

b) $D_t\left[t^3\ln(t+1)\right] = 3t^2\ln(t+1) + t^3\cdot\dfrac{1}{t+1} = 3t^2\ln(t+1) + \dfrac{t^3}{t+1}$

c) $D_x(8^x) = 8^x\ln 8$ **d)** $D_x(4^{x^2+1}) = 4^u\ln 4\dfrac{du}{dx} = 4^{x^2+1}2x\ln 4$

e) $D_x(2^{x^3e^{4x}}) = D_x(2^u) = 2^u\ln 2\dfrac{du}{dx} = 2^{x^3e^{4x}}(\ln 2)(3x^2e^x + x^3e^{4x}4) = 2^{x^3e^{4x}}(\ln 2)x^2e^{4x}(3+4x)$

4. a) Complete the basic log rules: $\ln(ab) = \ln a + \ln b$, $\ln(\frac{a}{b}) = \ln a - \ln b$, and $\ln(a^b) = b\ln a$.

b) $= D_x[\ln x - \ln x] = D_x[0] = 0$

c) $= D_x[\ln(2+\cos(x^2)) - \ln(x^8+1)] = \dfrac{1}{2+\cos(x^2)}\cdot(-\sin(x^2)\cdot 2x - \dfrac{1}{x^8+1}\cdot 8x^7 = -\dfrac{2x\sin(x^2)}{2+\cos(x^2)} - \dfrac{8x^7}{x^8+1}$

d) $= D_t[t\ln t] = \ln(t) + 1$

e) $= D_x[x\ln 9] = \ln 9$ Here $\ln 9$ is just a constant

f) The slope is $m = f'(1) = \ln(1) + 1 = 1$. The equation is $y - 0 = 1(x-1)$ or $y = x - 1$.

5. a) $\ln y = \ln(\sin 9x)^x = x\ln(\sin 9x) \implies \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln(\sin 9x) + \dfrac{x\cos(9x)\cdot 9}{\sin 9x} \implies \dfrac{dy}{dx} = (\sin 9x)^x(\ln(\sin 9x) + 9x\cot 9x)$.

b) $\ln y = \ln x^{2\sin x} = 2\sin x\ln x \implies \dfrac{1}{y}\cdot\dfrac{dy}{dx} = 2\cos x\ln x + (2\sin x)\dfrac{1}{x} \implies \dfrac{dy}{dx} = x^{2\sin x}\left(2\cos x\ln x + \dfrac{2\sin x}{x}\right)$.

c) $\ln y = \ln x^{\ln x} = \ln x\ln x = (\ln x)^2 \implies \dfrac{1}{y}\cdot\dfrac{dy}{dx} = 2(\ln x)\cdot\dfrac{1}{x} \implies \dfrac{dy}{dx} = x^{\ln x}\cdot\dfrac{2\ln x}{x}$.

d) $6^x\ln 6$. Logarithmic differentiation is not required.

6. For each relation, find $\frac{dy}{dx}$ by implicit differentiation.

a) $\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(2x) \implies 2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 2 \implies (x^2+2xy)\frac{dy}{dx} = 2 - 2xy - y^2 \implies \frac{dy}{dx} = \frac{2-2xy-y^2}{x^2+2xy}$

b) $\dfrac{d}{dx}[2x^3 - \sin(xy)] = \dfrac{d}{dx}(2) \implies 6x^2 - \cos(xy)\left(y + x\dfrac{dy}{dx}\right) = 0 \implies x\cos(xy)\dfrac{dy}{dx} = y\cos(xy) - 6x^2 \implies \dfrac{dy}{dx} = -\dfrac{y\cos(xy) - 6x^2}{x\cos(xy)}$

c) The point is $(-1,2)$. The slope is $\left.\dfrac{dy}{dx}\right|_{(-1,2)} = \dfrac{2 - 2(-1)(2) - 2^2}{(-1)^2 + 2(-1)(2)} = -\dfrac{2}{3}$. So the equation is $y - 2 = -\frac{2}{3}(x+1)$.

7. a) Let $y = \ln x$. We want to find $\frac{dy}{dx} = \frac{d}{dx}(\ln x)$. Start with

$$y = \ln x$$

Apply the inverse function of $\ln x$ which is the exponential function e^x.

$$e^y = e^{\ln x} = x.$$

b) Take the derivative using the chain rule on the left:

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \cdot \frac{dy}{dx} = 1.$$

c) Solve for $\frac{dy}{dx}$ to get: $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$.

8. **a)** $\dfrac{dy}{dx} = \dfrac{6x\ln(3x^2) - 3x^2\frac{6x}{3x^2}}{[\ln(3x^2)]^2} = \dfrac{6x[\ln(3x^2) - 1]}{[\ln(3x^2)]^2}$

b) $\dfrac{1}{\sin(e^{x^3+1})} \cdot \cos(e^{x^3+1}) \cdot (e^{x^3+1}) \cdot 3x^2 = 3x^2(e^{x^3+1})\cot(e^{x^3+1})$

c) $\ln y = \ln((\cos x)^{\tan(x^2)}) = \tan(x^2)\ln(\cos x) \implies \dfrac{1}{y} \cdot \dfrac{dy}{dx} = \sec^2 x \cdot \ln(\cos x) + \tan(x^2) \cdot \dfrac{-\sin x}{\cos x}$

$$\implies \frac{dy}{dx} = y\left(\sec^2 x \cdot \ln(\cos x) - \tan(x^2)\tan x\right) \implies \frac{dy}{dx} = (\cos x)^{\tan x}\left(\sec^2(x^2) \cdot \ln(\cos x) - \tan(x^2)\tan x\right)$$

d) $D_x(2^{x^2 \sin x}) = 2^u \frac{du}{dx} \ln 2 = 2^{x^2 \sin x}(2x\sin x + x^2 \cos x)\ln 2$

9. a) $g(x)$ is the inverse of $f(x)$ if $f(g(x)) = \underline{\ x\ }$ for all x in the domain of g and $g(f(x)) = \underline{\ x\ }$ for all x in the domain of f .

b) Let $f(x) = x^3 + 7$. This function consists of two operations on x, `cubing and adding 7`. In which order do they occur? `Cubing is first, then adding 7`. If you wanted to 'undo' f what operation would you have to do first? `Subtract 7`. Second? `Take the cube root`. Use the two 'undoing' operations in the correct order to determine the inverse function: $g(x) = \sqrt[3]{x-7}$.

c) First $f(g(x)) = f\left(\sqrt[3]{x-7}\right) = \left(\sqrt[3]{x-7}\right)^3 + 7 = x - 7 + 7 = x$. Next $g(f(x)) = g\left(x^3+7\right) = \sqrt[3]{(x^3+7)-7} = \sqrt[3]{x^3} = x$.

d) Draw a differentiable function that does NOT have an inverse. What can you say about the slope of such a function? `Your graph should be connected (continuous) and smooth (no corners, differentiable) AND it must fail the horizontal line test. Notice that the function must either go up and then down or down and then up to fail the horizontal line test. So the the slope of` f `is positive at some points and negative at others (and likely 0 in between).`

10. a) We say that $f(x)$ is **one-to-one** if whenever $x_1 \neq x_2$, then $f(x_1) \underline{\neq f(x_2)}$.

b) Let $f(x) = x^4 + 2x^2 + 3$. Show that f is NOT one-to-one by finding two different values x_1 and x_2 for which $f(x_1) = f(x_2)$. `Any pair of non-zero numbers of the form` a `and` $-a$ `work. For example,` $f(1) = 1 + 2 + 3 = 6$ `and` $f(-1) = 1 + 2 + 3 = 6$.

11. a) Let $y = e^x$. We want to find $\frac{dy}{dx}$. But $y = e^x \implies \ln y = \underline{\ln e^x} = \underline{x}$.

b) Take the derivative using the chain rule on the left: $\frac{d}{dx}[\ln y] = \frac{d}{dx}[x]$ or $\underline{\frac{1}{y} \cdot \frac{dy}{dx}} = \underline{1}$.

c) Solve for $\frac{dy}{dx}$ to get: $\frac{dy}{dx} = y = e^x$. Easy!!

Math 130, Lab 8 Quiz. Names: ______________________________

1. Determine $D_x \left[(6x^2+1)^{4^{\cos x}}\right]$

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