## Math 130: Lab 10

1. a) EVT: The Extreme Value Theorem. Assume that $f$ is a $\qquad$ function on the $\qquad$ interval $[a, b]$. Then $f$ has both an $\qquad$ and an on $[a, b]$.
b) MVT: The Mean Value Theorem] Assume that
2. $f$ is $\qquad$ on the $\qquad$ interval $[a, b]$;
3. $f$ is $\qquad$ on the $\qquad$ interval $(a, b)$;
Then there is some point $c$ in $(a, b)$ so that $f^{\prime}(c)=$ $\qquad$ .
4. a) Draw the graph of a function which is continuous on $[0,5]$ and which has an absolute max at $x=2$ and an absolute $\min$ at $x=4$ or indicate why this is impossible.
b) Draw the graph of a function on $[0,5]$ which has no absolute max or min or indicate why this is impossible.
c) Draw the graph of a continuous function on $(0,5)$ which has no absolute max or indicate why this is impossible.
d) Draw the graph of a function which is differentiable on $[0,5]$ which has no absolute max or indicate why this is impossible.
5. An icicle is in the shape of a right circular cone. At a certain point in time the height is 15 cm and is increasing at the rate of $1 \mathrm{~cm} / \mathrm{hr}$; the radius is 2 cm and decreasing at the rate $\frac{1}{10} \mathrm{~cm} / \mathrm{hr}$. Is the volume (!! you should know this: $V=\frac{1}{3} \pi r^{2} h$ ) increasing or decreasing at this instant? At what rate? (Ans: $-2 \pi / 3 \mathrm{cu} . \mathrm{cm} / \mathrm{hr}$ )
6. a) Determine the critical points of $f(x)=x^{3} e^{x}$.
b) Now determine the absolute extreme values of $f(x)$ on $[-2,1]$.
c) What theorem did you use to do this?

7. A 10 meter tall flag pole topples to the ground in such a way that the angle $\theta$ between the ground and the pole decreases at $0.12 \mathrm{rad} / \mathrm{s}$.
a) How is the distance $y$ between the tip of the flag pole and the ground changing when $y=8$ meters? Hint: Use a trig function. (Ans: $-0.72 \mathrm{~m} / \mathrm{s}$ )
b) How fast is the area of the triangle formed by the pole and the ground changing at this same moment? Is the area increasing or decreasing? Hint: Write each leg of the triangle using a trig function. (Ans: $1.68 \mathrm{~m}^{2} / \mathrm{s}$ )
8. Let $f(x)=\frac{2 x+4}{x^{2}+5}$. Determine the absolute extreme values of $f(x)$ on $[-2,3]$ and the points at which they occur.
9. a) Let $f(x)=x^{4}+4 x^{3}-20 x^{2}+1$. Find the critical points of $f$.
b) Determine the intervals where $f$ is increasing and where $f$ is decreasing.
c) What theorem did you use?
10. Determine where the absolute extrema of $y=x^{x}$ on the interval $[.2,1]$ occur. Hint: What type of differentiation must you use?
11. a) A problem from an exam in 2014. A baseball diamond is a square with 90 ft sides. Derek Jeter hits the ball and runs towards first base at a speed of $24 \mathrm{ft} / \mathrm{s}$. At what rate is his distance from third base changing when he is halfway to first base? (Answer: $\frac{24}{\sqrt{5}} \approx 10.733=\mathrm{ft} / \mathrm{s}$ )
b) How is the angle $\theta$ changing at this same moment? (Answer: $0.21 \overline{3} \mathrm{rad} / \mathrm{s}$.)
12. a) Determine the critical points of $f(x)=3\left(x^{2}-1\right)^{5 / 3}+2$.
b) Now determine the absolute extreme values of $f(x)$ on $[-2,2]$.
c) What theorem did you use to do this?

## Brief Answers

Complete answers online.

1. a) continuous function; closed interval; absolute maximum value; absolute minimum value.
b) continuous, closed; differentiable, open; $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$
2. $(\mathrm{a}-\mathrm{c})$ are possible; (d) is impossible.
3. a) Critical points: $x=0,-3$.
b) Absolute max at $x=1$ with $f(1)=e$ and absolute min at $x=-2$ with $f(-2)=-8 e^{-2}$.
c) Use the Closed Interval Theorem
4. Use the Closed Interval Theorem (CIT) on $[-2,3]$. Critical points: $x=-5,1$. But -5 is not in the interval. Absolute $\max$ at $x=1$ with $f(1)=1$ and absolute min at $x=-2$ with $f(-2)=0$.
5. a) Find the derivative and the critical points $\left(f^{\prime}(x)=0\right.$ or DNE).

$$
f^{\prime}(x)=4 x^{3}+12 x^{2}-40 x=4 x\left(x^{2}+3 x-10\right)=4 x(x+5)(x-2)=0 \quad \text { at } \quad x=-5,02 .
$$

Now record this information on a number line for easy reference.

b) Use the Increasing/Decreasing Test. Determine the sign of $f^{\prime}(x)$ between and beyond the critical points. Here we use the IVT to know that the only places that the derivative can change sign are at the critical points because the derivative is continuous. Just plug in values in the appropriate intervals. $f^{\prime}(-6)=-192, f^{\prime}(-1)=+48$, $f^{\prime}(1)=-24$, and $f^{\prime}(3)=+96$.


Using interval notation: $f$ is increasing on $(-5,0)$ and $(2, \infty)$ and it is decreasing on $(-\infty,-5)$ and $(0,2)$.
8. Use the Closed Interval Theorem (CIT) on $[.2,1]$. Absolute max at $x=1$ with $f(1)=$ and absolute min at $x=-e^{-1}$ with $f\left(e^{-1}\right)=\left(e^{-1}\right)^{e^{-1}} \approx 0.6922006$.
10. a) Use the Closed Interval Theorem on $[-2] 2$,$] . Critical points: x=-1,0,1$.

- At the critical points: $f(-1)=3(0)^{5 / 3}+2=2, f(0)=3(-1)+2=-1, f(1)=2$.
- At the endpoints: $f(-2)=3(3)^{5 / 3}+2 \approx 20.72$ and $f(2) \approx 20.72$.
- Absolute $\max \approx 20.72$ at $x=2$ and -2 and absolute $\min$ at $x=0$ with $f(0)=1$.


## Answers to Lab 10

1. a) EVT: The Extreme Value Theorem. Assume that $f$ is a continuous function function on the closed interval $[a, b]$. Then $f$ has both an absolute max and an absolute min on $[a, b]$.
b) MVT: The Mean Value Theorem] Assume that
2. $f$ is continuous on the closed interval $[a, b]$;
3. $f$ is differentiable on the open interval $(a, b)$;

Then there is some point $c$ in $(a, b)$ so that $f^{\prime}(c)=\frac{f(b)-f(a)}{b-a}$.
2. a) Possible: See figure below. In all of these problems make sure $f$ is defined at ALL points in the given interval.


Part (a)


Part (b)


Part (c)
b) Possible. The function need not be continuous. See above. Make sure $f$ is defined at each point in $[0,5]$.
c) Possible. See the graph above.
d) Impossible: If $f$ is differentiable, then $f$ is continuous. The EVT says a continuous function on $[0,5]$ must have an absolute max and an absolute min.
3. Given $\frac{d h}{d t}=1 \mathrm{~cm} / \mathrm{hr}$ and $\frac{d r}{d t}=-0.1 \mathrm{~cm} / \mathrm{hr}$. Find $\left.\frac{d V}{d t}\right|_{h=15, r=2}$. Relation: $V=\frac{1}{3} \pi r^{2} h$. Rate-ify: $\frac{d V}{d t}=\frac{2 \pi}{3} r h \frac{d r}{d t}+\frac{\pi}{3} r^{2} \frac{d h}{d t}$. So

$$
\left.\frac{d V}{d t}\right|_{h=15, r=2}=\frac{2 \pi}{3}(2)(15)(-0.1)+\frac{\pi}{3}(2)^{2}(1)=-\frac{2 \pi}{3} \mathrm{~cm}^{3} / \mathrm{hr}
$$

4. a) Critical points: $f^{\prime}(x)=3 x^{2} e^{x}+x^{3} e^{x}=\left(x^{3}+3 x^{2}\right) e^{x}=x^{2}(x+3) e^{x}=0$. So $x=0,-3$.
b) Use the Closed Interval Theorem (CIT) on $[-2,1]$.

- At the critical points: $f(0)=0$. Note -3 is NOT in the in the interval.
- At the endpoints: $f(-2)=-8 e^{-2} \approx-1.082682$ and $f(1)=e \approx 2.2718$.
- Absolute max at $x=1$ with $f(1)=e$ and absolute min at $x=-2$ with $f(-2)=-8 e^{-2}$.

5. a) Given $\frac{d \theta}{d t}=-0.12 \mathrm{rad} / \mathrm{s}$. Find $\left.\frac{d y}{d t}\right|_{y=8}$. Relation: $\frac{y}{10}=\sin \theta$ or $y=10 \sin \theta$. Rate-ify: $\frac{d y}{d t}=10 \cos \theta \frac{d \theta}{d t}$. When $y=8$, $x=\sqrt{10^{2}-8^{2}}=6$ and so $\cos \theta=\frac{6}{10}=0.6$. Thus,

$$
\frac{d y}{d t}=10(0.6)(-0.12)=-0.72 \mathrm{~m} / \mathrm{s}
$$

b) Given $\frac{d \theta}{d t}=-0.12 \mathrm{rad} / \mathrm{s}$. Find $\left.\frac{d A}{d t}\right|_{y=8}$. Note: $y=10 \sin \theta$ and the base $x=10 \cos \theta$. Relation: $A=\frac{1}{2} x y=\frac{y}{10}=$ $\frac{1}{2}(10 \sin \theta)(10 \cos \theta)=50 \sin \theta \cos \theta$. Relation: $\frac{d A}{d t}=50 \cos ^{2} \theta \frac{d \theta}{d t}-50 \sin ^{2} \theta \frac{d \theta}{d t}$. When $y=8, \cos \theta=\frac{6}{10}=0.6$ and $\sin \theta=\frac{8}{10}=0.8$. So

$$
\frac{d A}{d t}=50(0.6)^{2}(-0.12)-50(0.8)^{2}(-0.12)=1.68 \mathrm{~m}^{2} / \mathrm{s}
$$

The area is increasing.
6. a) Use the Closed Interval Theorem (CIT) on $[-2,3]$. Critical points:

$$
f^{\prime}(x)=\frac{2\left(x^{2}+5\right)-(2 x+4)(2 x)}{\left(x^{2}+5\right)^{2}}=\frac{-2 x^{2}-8 x+10}{\left(x^{2}+5\right)^{2}}=\frac{-2\left(x^{2}+4 x-5\right)}{\left(x^{2}+5\right)^{2}}=\frac{-2(x+5)(x-1)}{\left(x^{2}+5\right)^{2}}=0
$$

at $x=-5,1$.

- At the critical points: $f(1)=1$. Note -5 is not in the interval.
- At the endpoints: $f(-2)=0$ and $f(3)=5 / 7$.
- Absolute max at $x=1$ with $f(1)=1$ and absolute min at $x=-2$ with $f(-2)=0$.

7. a) Find the derivative and the critical points $\left(f^{\prime}(x)=0\right.$ or DNE).

$$
f^{\prime}(x)=4 x^{3}+12 x^{2}-40 x=4 x\left(x^{2}+3 x-10\right)=4 x(x+5)(x-2)=0 \quad \text { at } \quad x=-5,02
$$

Now record this information on a number line for easy reference.

b) Use the Increasing/Decreasing Test. Determine the sign of $f^{\prime}(x)$ between and beyond the critical points. Here we use the IVT to know that the only places that the derivative can change sign are at the critical points because the derivative is continuous. Just plug in values in the appropriate intervals. $f^{\prime}(-6)=-192, f^{\prime}(-1)=+48$, $f^{\prime}(1)=-24$, and $f^{\prime}(3)=+96$.


Using interval notation: $f$ is increasing on $(-5,0)$ and $(2, \infty)$ and it is decreasing on $(-\infty,-5)$ and $(0,2)$.
8. Use Logarithmic Differentiation. Critical points: $\ln y=\ln x^{x}=x \ln x$. So

$$
\frac{1}{y} \frac{d y}{d x}=\ln x+x \cdot 1 x=\ln (x)+1 \quad \text { or } \quad \frac{d y}{d x}=x^{x}(\ln (x)+1)=0
$$

Now $x^{x}$ is never 0 so set $\ln (x)+1=0$. Then

$$
\ln x=-1 \quad \text { or } \quad x=e^{-1}
$$

Use the Closed Interval Theorem (CIT) on $[0.2,1]$.

- At the critical point: $f\left(e^{-1}\right)=\left(e^{-1}\right)^{e^{-1}} \approx 0.6922006$.
- At the endpoints: $f(0.2)=(0.2)^{0.2} \approx 0.72477$ and $f(1)=1^{1}=1$.
- Absolute max at $x=1$ with $f(1)=$ and absolute min at $x=e^{-1}$ with $f\left(e^{-1}\right)=\left(e^{-1}\right)^{e^{-1}} \approx 0.6922006$.

9. a) Given $\frac{d a}{d t}=24 \mathrm{ft} / \mathrm{s}$. Find $\left.\frac{d c}{d t}\right|_{a=45}$. Relation: $c^{2}=a^{2}+90^{2}$. Rate-ify: $2 c \frac{d c}{d t}=2 a \frac{d a}{d t}$ or $\frac{d c}{d t}=\frac{a}{c} \frac{d a}{d t}$. When $a=45$, $c=\sqrt{45^{2}+90^{2}}=45 \sqrt{5}$. So

$$
\frac{d c}{d t}=\frac{45}{45 \sqrt{5}}(24)=\frac{24}{\sqrt{5}} \approx 10.733=\mathrm{ft} / \mathrm{s}
$$

b) Now find $\left.\frac{d \theta}{d t}\right|_{a=45}$. Relation (use the constant $\operatorname{side!}$ ): $\tan \theta=\frac{a}{90}$, so $\theta=\arctan \left(\frac{a}{90}\right)$. Rate-ify:

$$
\frac{d \theta}{d t}=\frac{1}{1+\left(\frac{a}{90}\right)^{2}} \cdot \frac{1}{90} \frac{d a}{d t}=\frac{90}{90^{2}+a^{2}} \frac{d a}{d t}
$$

When $a=45$,

$$
\left.\frac{d \theta}{d t}\right|_{a=45}=\frac{90}{90^{2}+45^{2}}(24)=0.21 \overline{3} \mathrm{rad} / \mathrm{s}
$$

b) Another method: find $\left.\frac{d \theta}{d t}\right|_{a=45} \cdot \tan \theta=\frac{a}{90}$. Rate-ify: $\sec ^{2} \theta \frac{d \theta}{d t}=\frac{1}{90} \frac{d a}{d t}$ or $\frac{d \theta}{d t}=\frac{1}{90 \sec ^{2} \theta} \frac{d a}{d t}=\frac{\cos ^{2} \theta}{90} \frac{d a}{d t}$. Use the triangle: When $a=45$, from part (a) we have $c=45 \sqrt{5}$, so $\cos ^{2} \theta=\left(\frac{90}{c}\right)^{2}=\left(\frac{90}{45 \sqrt{5}}\right)^{2}=\left(\frac{2}{\sqrt{5}}\right)^{2}=\frac{4}{5}=0.8$. So

$$
\left.\frac{d \theta}{d t}\right|_{a=45}=\frac{\cos ^{2} \theta}{90} \frac{d a}{d t}=\frac{0.8}{90}(24)=0.21 \overline{3} \mathrm{rad} / \mathrm{s}
$$

10. a) Critical points: $f^{\prime}(x)=5\left(x^{2}-1\right)^{2 / 3} 2 x=10 x[(x-1)(x+1)]^{2 / 3}=0$. So $x=-1,0,1$.
b) Use the Closed Interval Theorem (CIT) on $[-2,2]$.

- At the critical points: $f(-1)=3(0)^{5 / 3}+2=2, f(0)=3(-1)+2=-1, f(1)=2$.
- At the endpoints: $f(-2)=3(3)^{5 / 3}+2 \approx 20.72$ and $f(2) \approx 20.72$.
- Absolute $\max \approx 20.72$ at $x=2$ and -2 and absolute min at $x=0$ with $f(0)=1$.

1. a) (8 pts) Let $f(x)=\sqrt{x^{4}-2 x^{2}+3}$ on $[-2,3]$. WeBWorK Day $28 \# 8$. Find the absolute max and min of $f$ and the points at which these occur. (You may assume the term in the square root is always positive.) Carefully simplify $f^{\prime}$. Show your work.

The absolute max value is $\qquad$ occurring at $x=$ $\qquad$ . The absolute min value is $\qquad$ occurring at $x=$ $\qquad$ .
b) What theorem did you use? $\qquad$
2. ( 6 pts ) From the next exam: Complete the statement of the Mean Value Theorem then draw and label diagram which illustrates it on the axes provided. Assume that $f$ is a $\qquad$ function on the $\qquad$ interval _a, $b_{\text {_ }}$ and $a$ $\qquad$ function on the $\qquad$ interval __ $a, b_{\_}$. Then there is a point $c$ in $(a, b)$ so that (fill in below)
$\qquad$ $=$ $\qquad$
a) (2 pts) Complete the definition. If $f$ is defined at $x=c$, then $c$ is a critical point of $f$ if:
3. Optional Bonus: Re-use some of the information in Problem 1 to determine the intervals where $f(x)=\sqrt{x^{4}-2 x^{2}+3}$ is increasing and where it is decreasing. Use a number line to summarize your results.

