# Math 130, Lab 12 and Practice For Test 3

1. Draw a detailed graph of the function $f(x) = \dfrac{x^2 + 1}{(x+1)^2}$. Include all extrema, inflections, and VAs and HAs. Indicate any intercepts as well. **Show all work.** Use an appropriate scale for each axis. To speed this up, here are the simplified derivatives that you may use. Create number lines for them; be careful at points where the original function is not define.

Domain, HAs, and VAs. (Use appropriate limits.)

$$f'(x) = \frac{2x(x+1)^2 - (x^2+1)(2)(x+1)}{(x+1)^4} = \frac{2x - 2}{(x+1)^3}.$$

$$f''(x) = \frac{2(x+1)^3 - (2x-2)(3)(x+1)^2}{(x+1)^6} = \frac{-4x + 8}{(x+1)^4}$$
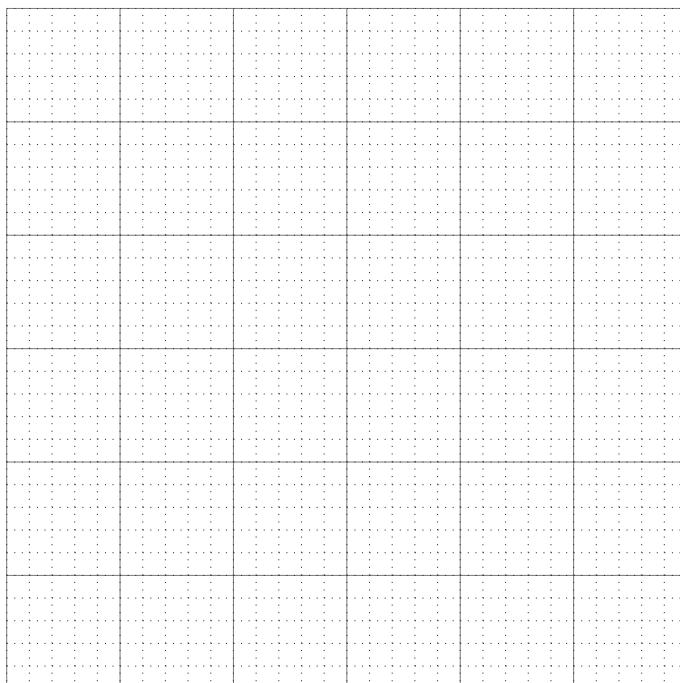
CIT: The Closed Interval Theorem. Let f be a continuous function on a closed interval $[a, b]$. Then the absolute extrema of f occur either at critical points of f on the open interval (a, b) or at the endpoints a and/or b.

SCPT: The Single Critical Point Theorem. Assume that f is a continuous function on an interval (open, closed, or half-open) I. Assume that c is the only critical number of f.

    a) If f has a relative maximum at c, then f has a absolute maximum at c.

    b) If f has a relative minimum at c, then f has a absolute minimum at c.

2. A cylindrical waste basket (no top) has a volume of 1000π cubic inches. What radius will minimize the material (surface area) of the basket.
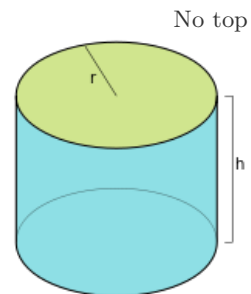
No top

Did you determine a domain for your function? Have you justified your answer based on this domain? (Ans: $r = 10$ in.)

3. A child's sandbox is to be made by cutting equal squares from the corners of a square sheet of galvanized aluminum and turning up the sides. If each side of the sheet of galvanized iron is 2 meters long, what size squares should be cut from the corners to maximize the volume of the sandbox?
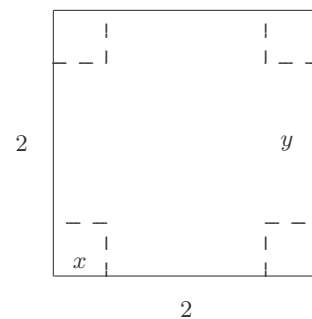
$$2 \qquad\qquad\qquad y$$

$$x$$

$$2$$

Did you determine a domain for your function? Have you justified your answer based on this domain? (Ans: $x = 1/3$ m).

4. From the Practice Test: Do a complete graph of $f(x) = \frac{1}{5}x^5 - \frac{4}{3}x^3$. (To receive full credit on the test, you must indicate critical points, relative extrema, increasing, decreasing, inflections, and concavity. Use number lines for the derivatives.) Make sure that you properly illustrate the increasing and decreasing behavior and the concavity of the function. WARNING: Messy values: Use your calculator to compute function values.
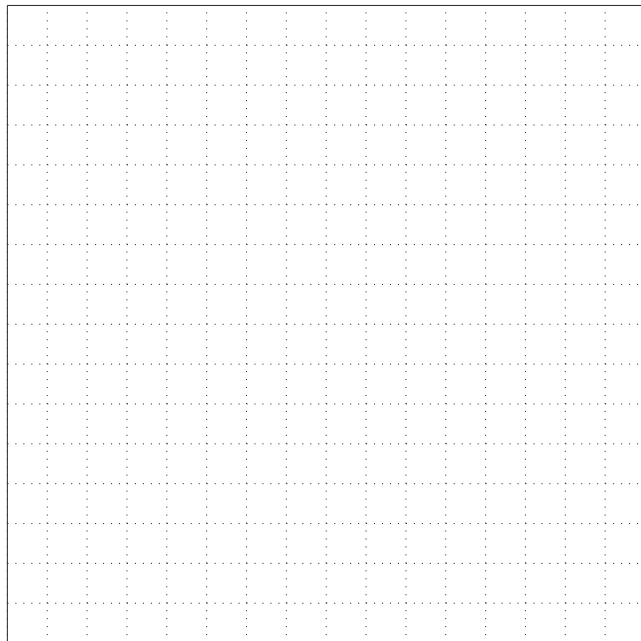
5. The graph below is the graph of $f'(x)$. Draw a possible graph of the original function $y = f(x)$. Determine where there are critical points, relative extremes, points of inflection, where the function is increasing, decreasing, concave up, and concave down. Use number lines to help organize your information.

a) Where is $f'(x) = 0$? Positive? Negative? Mark this on the number line.

b) Where is $f''(x) = 0$? (You can answer this question by remembering that f'' is the derivative (slope) of f'. So where is the *slope* of the function below 0? Where is it positive? Negative? Mark this on the number line.

c) Now draw the graph of f start at the dot at $(-5, 4)$

This f', NOT the original function f

f'

$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

f''

$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$
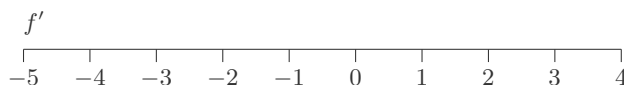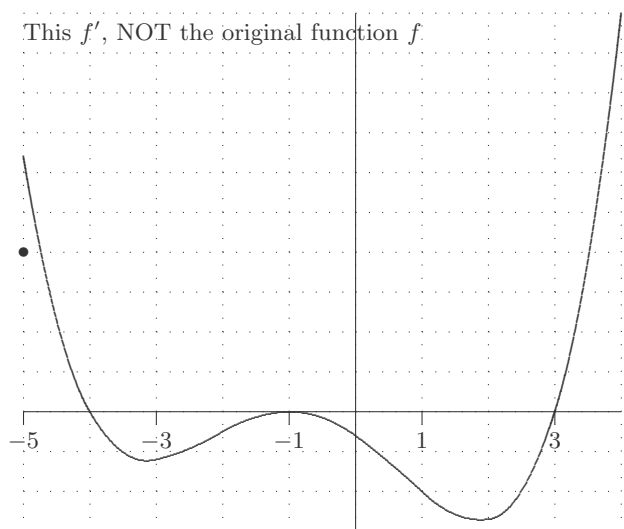
6. President Gearan recently announced some 'improvements' to the football field. It will be turned into an equestrian center. He will build a corral in front of a 100-foot long stable (see below) using 180 feet of fence $(+ + +)$. What dimensions for the corral maximize the area? (He does not need fence along the front of the stable and the corral must extend the entire length of the stable.) (Ans: 4000 sq. ft.)
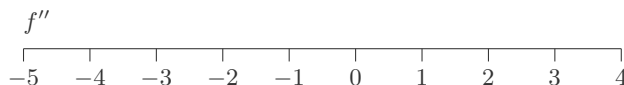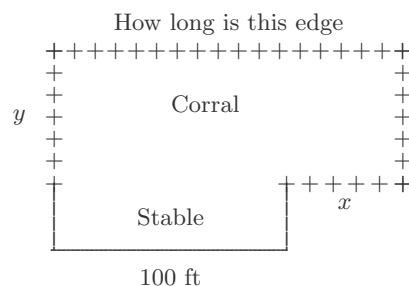
How long is this edge

y Corral

Stable

x

100 ft

Did you determine a domain for your function? Have you justified your answer based on this domain?

5

7. (Alfred Croteau (S' 04)): A rectangle has a diagonal of 2 meters. What dimensions of the rectangle produce the largest area? Justify your answer.
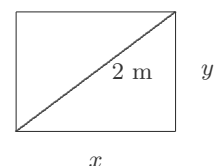
Did you determine a domain for your function? Have you justified your answer based on this domain?

8. Below I give you information about the derivatives of the *continuous* function g. First determine where the function is increasing and decreasing. Next classify each critical point as a relative max, relative min, or neither. T hen determine where the function is concave up and down and the location of any inflections. Sketch the graph function that would have derivatives like those given. Indicate on your graph which points are local extrema. DNE indicates that the derivative does not exist at the point (though the original function does). You will need to make up function values for the critical points that are consistent with the given information. The only point you know the function passes through is $(0,0)$, which is marked on the graph.

a)

g'

$$-- \quad \text{DNE} \quad ++ \quad 0 \quad --$$
$$\quad -1 \phantom{\text{DNE}} \quad 2$$

g''

$$++ \quad \text{DNE} \quad -- \quad 0 \quad ++$$
$$\quad -1 \phantom{\text{DNE}} \quad 3$$
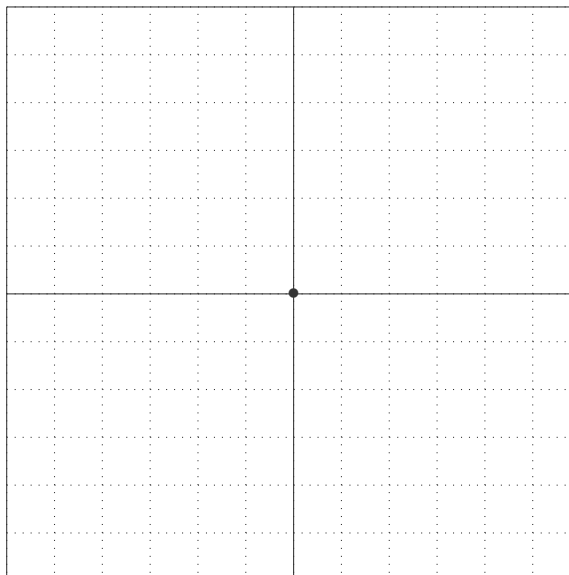
Math 130, Lab 12 Bonus. Names: _____

9. Optical fiber from a computer lab to a physics lab is being laid. The physics lab lies across a 15 m wide road and is 25 m down the other side from the computer lab. (See figure.) It costs \$13(00) per meter to put cable under the road and \$12(00) per meter to bury it underground. Where should the cable be brought across the road?

Phys Lab • – – – – – – – – – – – – – – – –

15 m y Road

                  • – – – • – – – – – – • – – – –
                  Q x P CS Lab

                  ———————— 25 ————————

Did you determine a domain for your function? Have you justified your answer based on this domain?

Math 130: Answers Lab 12

1. Domain: $(-\infty, -1) \cup (-1, +\infty)$; all $x \neq -1$.

VA: $\displaystyle\lim_{x \to -1^+} \frac{\overbrace{x^2+1}^{2}}{\underbrace{(x+1)^2}_{0^+}} = +\infty$ and $\displaystyle\lim_{x \to -1^-} \frac{\overbrace{x^2+1}^{2}}{\underbrace{(x+1)^2}_{0^+}} = +\infty$; so VA at $x = -1$.

HA: Using l'Hopital: $\displaystyle\lim_{x \to +\infty} \frac{x^2+1}{(x+1)^2} = \lim_{x \to +\infty} \frac{2x}{2(x+1)} = \lim_{x \to +\infty} \frac{2}{2} = 1$ so HA at $y = 1$.

HA: $\displaystyle\lim_{x \to -\infty} \frac{x^2+1}{(x+1)^2} = \lim_{x \to -\infty} \frac{2x}{2(x+1)} = \lim_{x \to -\infty} \frac{2}{2} = 1$ so HA at $y = 1$.

$f'(x) = \dfrac{2x(x+1)^2 - (x^2+1)(2)(x+1)}{(x+1)^4} = \dfrac{2x(x+1) - (x^2+1)(2)}{(x+1)^3} = \dfrac{2x-2}{(x+1)^3} = 0$ at $x = 1$; $x = -1$ not in domain (NID)

$f''(x) = \dfrac{2(x+1)^3 - (2x-2)(3)(x+1)^2}{(x+1)^6} = \dfrac{2(x+1) - (2x-2)(3)}{(x+1)^4} = \dfrac{-4x+8}{(x+1)^4} = 0$ at $x = 2$; $x = -1$ NID.

Evaluate f at key points. $f(1) = \frac{1}{2}$, $f(2) = \frac{5}{9}$, intercept $f(0) = 1$, and another when $x < 0$: $f(-3) = \frac{5}{2}$

		RMin		
f' ++	NID −−	0	++	
Inc	-1 Dec	1	Inc	

		INF		
f'' ++	NID ++	0	−−	
CU	-1 CU	2	CD	

Increasing | Dec | Increasing

RMin $(1, .5)$
Infl $(2, .555)$
HA: $y = 1$
VA: $x = -1$
Concave UP | C UP | Concave DN

2.

Maximize:	Surface $S = \pi r^2 + 2\pi rh$ (bottom circle plus side)		
Constraint:	Volume $V = 1000\pi = \pi r^2 h$		
Eliminate:	h. $h = \frac{1000}{r^2}$	Domain: $(0, \infty)$.	
Substitute:	$S = \pi r^2 + (2\pi r)\frac{1000}{r^2} = \pi r^2 + \frac{1000\pi}{r}$	on $(0, \infty)$	
Differentiate:	$S' = 2\pi r - \frac{2000\pi}{r^2} = 0 \implies 2\pi r = \frac{2000\pi}{r^2} \implies r^3 = 1000,\ r = 10$		
Justify:	Use SCPT. Find the critical pts.	Classify the critical number.	

1st derivative test: S' $\dfrac{\quad -\,-\,- \quad 0 \ +\,+\,+ \quad}{\overset{\mid}{10}}$ Rel Min at $r = 10$

Rel min at $r = 10$ is Abs Min by SCPT.

---

3. Use the basic format.

Maximize:	Volume $V = xy^2$	
Constraint:	Sheet Edge $2 = 2x + y$ where $0 \leq x \leq 1$.	
Eliminate y:	$y = 2 - 2x$.	
Substitute:	$V(x) = x(2-2x)^2 = 4x - 8x^2 + 4x^3$	Domain $[0, 1]$.
Differentiate:	$V'(x) = 4 - 16x + 12x^2 = 4(1 - 4x + 3x^2) = 4(1-x)(1-3x) = 0$	$x = 1,\ 1/3$
Justify:	CIT. Crit pt: $V(1/3) = \frac{1}{3}(\frac{4}{3})^2 = \frac{16}{27}$	$V(0) = 0 = V(1)$
	CIT: At $x = 1/3$ and $V(1/3) = \frac{16}{27}\text{m}^3$ is Abs Max.	

---

4. Graph $f(x) = \frac{1}{5}x^5 - \frac{4}{3}x^3$.

r max $(-2, 64/15)$

infl $(-\sqrt{2}, 2.64)$

infl $(0,0)$

-2　　　　　　0　　　　2

f

infl $(\sqrt{2}, -2.64)$

r min $(2, -64/15)$

Locate critical numbers:

$f'(x) = x^4 - 4x^2 = x^2(x^2 - 4)$ at $x = 0, \pm 2$.

	inc	rmax	dec	Neither	dec	rmin	inc
f'	$++$	0	$---$	0	$---$	0	$++$

$$-2 \qquad 0 \qquad 2$$

Check concavity with the second derivative.

$f''(x) = 4x^3 - 8x = 4x(x^2 - 2)$　at　$x = \pm\sqrt{2}$.

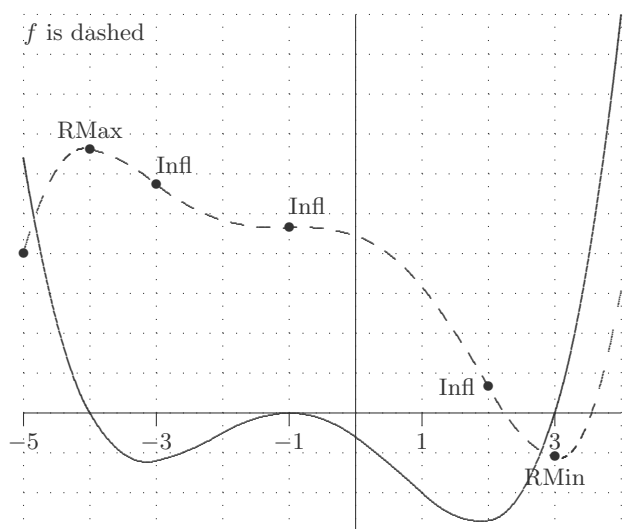|	concave dn	inf	up	inf	dn	inf	concave up
f''	$---$	0	$++$	0	$--$	0	$+++$

$$-(2)^{1/2} \qquad 0 \qquad (2)^{1/2}$$

Plot the critical numbers and the inflections:

$f(0) = 0$ and $f(2) = -64/15$, $f(-2) = 64/15$.

$f(\sqrt{2}) \approx -2.64$ and $f(-\sqrt{2}) \approx 2.64$.

5. The graph below is the graph of $f'(x)$. Draw a possible graph of the original function $y = f(x)$.

f is dashed

RMax

Infl

Infl

Infl

-5　　　-3　　　-1　　　1　　　3

RMin

	Inc	RMx	Dec		CP		Dec		RMn	Inc
$f'+$	0		$---$		0		$---$		0	$++$

$$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

	C Dn	Infl	C Up	Infl		C Dn		Infl	C Up
f''	$---$	0	$++$	0		$---$		0	$++$

$$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$$

6.

Maximize:　　Area $A = xy$

Constraint:　　$2 = \sqrt{x^2 + y^2}$

Eliminate:　　$y^2 = 4 - x^2$　　　　　　$y = \sqrt{4 - x^2}$　　　　　　　　Domain: $0 \le x \le 2$

Substitute:　　$A = x\sqrt{4 - x^2}$　　　　$[0, 2]$

Differentiate:　$A' = \sqrt{4 - x^2} + x\frac{(-2x)}{2\sqrt{4-x^2}}$　　So $4 - x^2 - x^2 = 0 \Rightarrow x = \pm\sqrt{2}$　　　Only $\sqrt{2}$ in domain

Justify:　　CIT Endpts: $A(0) = 0$, $A(2) = 0$　Crit #: $A(\sqrt{2}) = 2$　　　Abs Max at $x = \sqrt{2}$ and $y = \sqrt{2}$.

7.

Maximize:	Area $A = (100 + x)y$.
Constraint:	Fence $F = x + (x + 100) + 2y = 180$
Eliminate:	$2y = 80 - 2x \Rightarrow y = 40 - x$
Substitute:	$A = (100 + x)(40 - x) = 4000 - 100x + 40x - x^2 = 4000 - 60x - x^2$
Differentiate:	$A' = -60 - 2x = 0 \Rightarrow x = -30$
Justify:	CIT Endpts: $A(0) = 100(40) = 4000$, $A(40) = 140(0) = 0$

$x + 100$ is top edge.
Domain: $0 \le x \le 40$
$[0, 40]$
Not in Domain!
Abs Max at $x = 0$, $y = 40$

8. There's a local min (and a corner) at $x = -1$, a local max at $x = 2$ and an inflection at $x = 3$. Make sure to get the concavity right!

g'

R. Min		R. Max	
$--$	DNE $++$	0	$--$
decr	-1 incr	2	decr

g''

Inflect		Inflect	
$+++$	$*$	$---$	$0 +$
con up	-1	con dn	3 c up

---
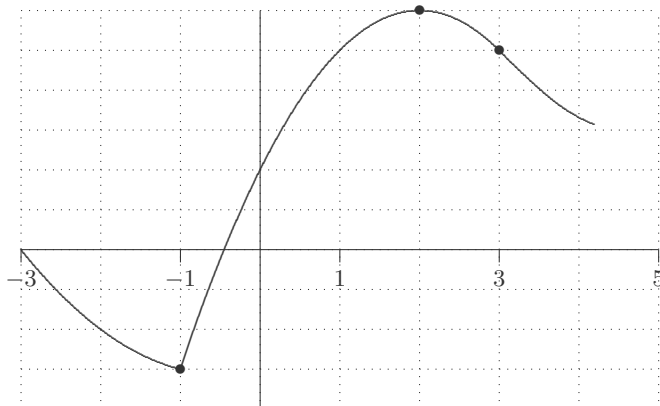
9. Use the basic procedure

1. Minimize the cost $C = 13y + 12(25 - x)$

2. Subject to $y^2 = x^2 + (15)^2$.

3. Eliminate y. $y = \sqrt{x^2 + 225}$ where $0 \le x \le 25$.

4. Rewrite C. $C(x) = 13\sqrt{x^2 + 225} + 12(25 - x)$ on $[0, 25]$.

5. Use CIT (or possibly SCPT). Find the critical numbers.

$$C'(x) = \frac{13x}{\sqrt{x^2 + 225}} - 12 = 0 \Rightarrow 13x = 12\sqrt{x^2 + 225} \Rightarrow 169x^2 = 144(x^2 + 225).$$

This means

$$25x^2 = (144)(225) \Rightarrow 5x = \pm 12(15) \Rightarrow x = \pm 36.$$

Since NEITHER point is in the interval. Use the CIT. The min occurs at an endpoint.
$C(0) = 13(15) + 12(25) = 495$.
$C(25) = 13\sqrt{625 + 225} = 379.01$.
So by the CIT, the absolute min occurs when $x = 25$ and the cost is \$379.01(00)=\$37,901.