Math 130: Lab XIII

1. (Review. Answers at bottom of page.) Determine these limits using HP. Remember $\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$.

a) $\displaystyle\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+1}}$

b) $\displaystyle\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+1}}$

c) $\displaystyle\lim_{x\to-\infty} \frac{|x|}{2x+1}$

Now use L'Hopital's rule to determine each of these limits after determining the indeterminate form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) of the limit.

d) $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{2x}$

e) $\displaystyle\lim_{x\to+\infty} \frac{\ln x^2}{x^2}$

f) $\displaystyle\lim_{x\to 0} \frac{5x-\sin 5x}{2x-\sin 2x} =$

2. Calculate these interesting limits. First classify each as indeterminate (indicate what type: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, 1^∞, ∞^0, or 0^0) or not. Answers not in order: 0, 0, 0, 1, 2, e^2, e^k, $+\infty$, $-\infty$

a) $\displaystyle\lim_{x\to 0^+} 2x\ln x$ **b)** $\displaystyle\lim_{x\to 0^+} (3x)^x$ **c)** $\displaystyle\lim_{x\to 0^+} (1+2x)^{1/x}$

d) $\displaystyle\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^x$ **e)** $\displaystyle\lim_{x\to 0} \frac{\arctan 4x}{\sin 2x}$ **f)** $\displaystyle\lim_{x\to 0} \frac{\sin 4x}{3\sec x}$

g) $\displaystyle\lim_{x\to 1^+} \frac{x^2+1}{1-x}$ **h)** $\displaystyle\lim_{x\to 0^+} \frac{\cos x}{x^2}$ **i)** $\displaystyle\lim_{x\to\infty} \frac{x\ln x}{x^2+1}$ **j)** Ans #1 : $\frac{3}{2}$, $-\frac{3}{2}$, $-\frac{1}{2}$, , 0, 0, $\frac{125}{8}$

GRAPHING NOTES: Remember the basic 5 steps: (1) Domain; (2) VAs; (3) HAs; (4) First derivative info; (5) Second derivative info. When finding HAs: determine both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For VAs at a: determine both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$. Make sure to evaluate f at all critical and inflection points. Make sure to **plot and label the actual key points of the function**.

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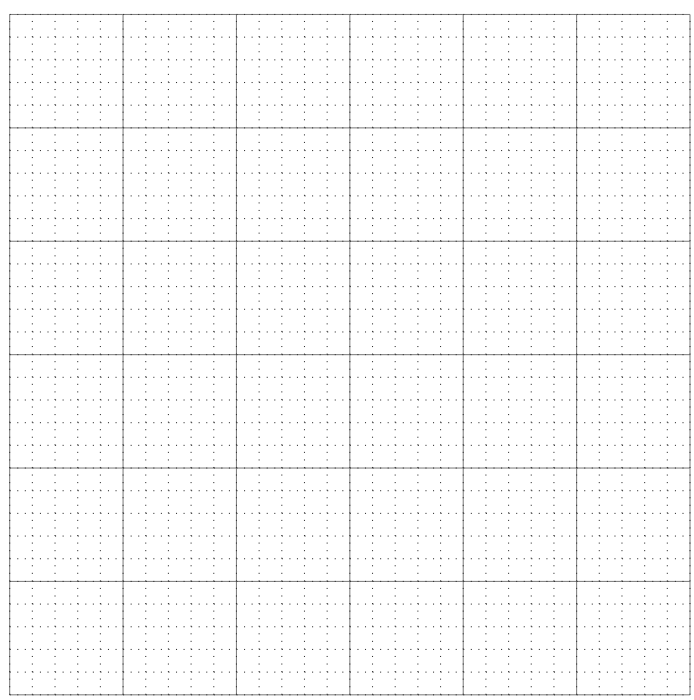
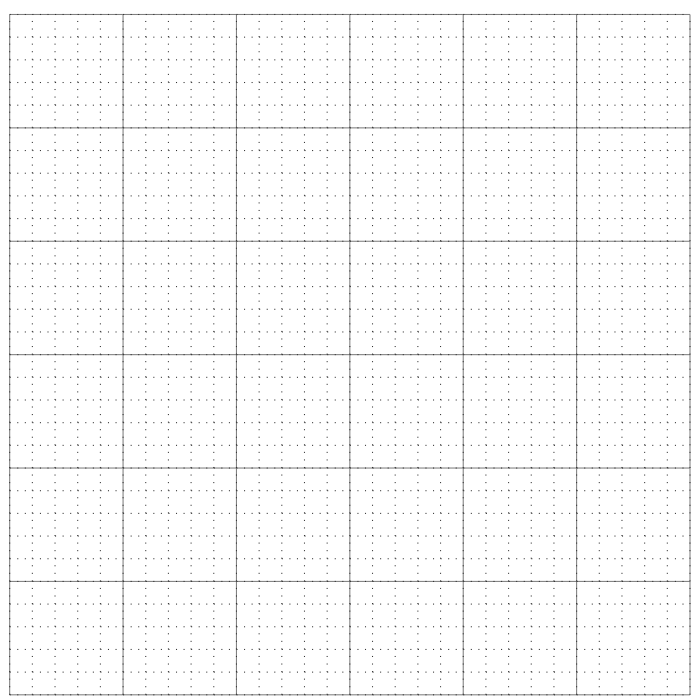
3. Draw a detailed graph of $y = f(x) = \dfrac{x}{x^2+1}$. To speed things up, you may assume: $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ and $f''(x) = \dfrac{2x^3-6x}{(x^2+1)^3}$.



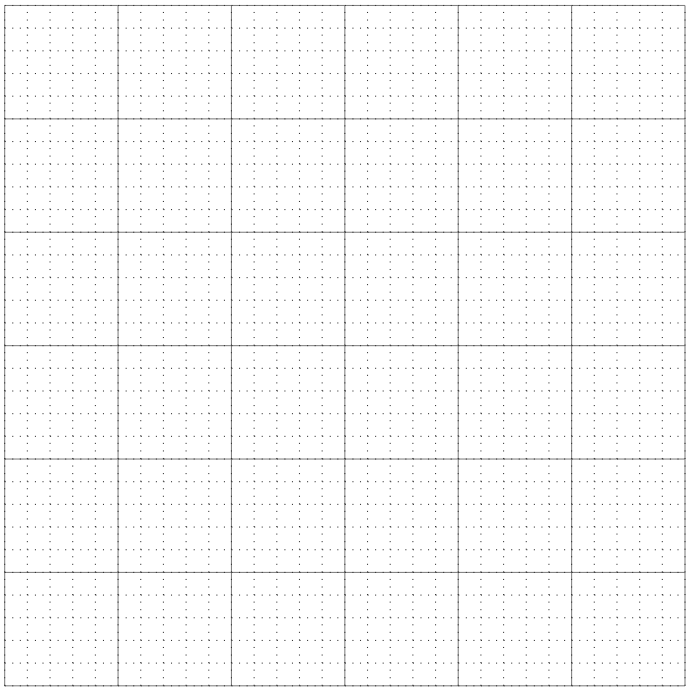
4. Here is information about the first and second derivatives of a function and its vertical and horizontal asymptotes. Sketch a function that satisfies these conditions. Indicate on your graph which points are local extrema and which are inflections. **NID** means the point is "not in the domain" of the function. Let $f(0) = 0$ and $\lim_{x\to 2^+} f(x) = +\infty$, $\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to +\infty} f(x) = 1$, and $\lim_{x\to -\infty} f(x) = +\infty$.

f'	$----$	0	$--$	NID	$--$	
		0		2		

f''	$+++$	0	$--$	NID	$++$	
		0		2		

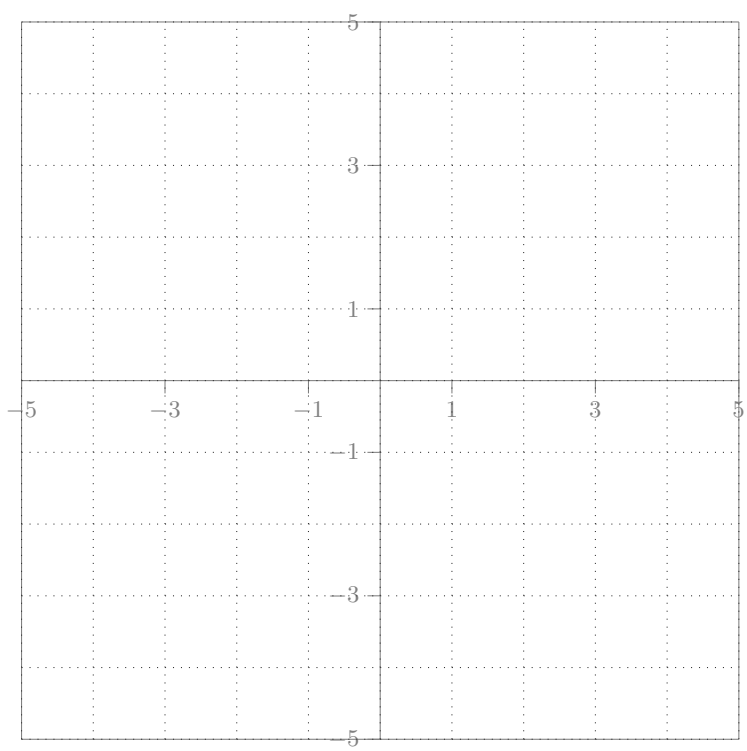


5. Graphing Practice. The function $f(x) = \dfrac{2(x+1)^2}{x^2}$ has derivatives $f'(x) = \dfrac{-4x-4}{x^3}$ and $f''(x) = \dfrac{8x+12}{x^4}$. Graph f paying special attention to critical points and points of inflection. Determine where there are **vertical** and **horizontal asymptotes**. Make sure that your graph properly illustrates the increasing and decreasing behavior and the concavity of the function.



6. Bonus. Let $f(x) = \dfrac{1}{e^x - 1}$. Graph f paying special attention to **vertical** and **horizontal asymptotes**. Make sure that your graph properly illustrates the increasing and decreasing behavior and the concavity of the function. Show your work! (But see hint below.)

You may use $f''(x) = \dfrac{e^{2x} + e^x}{(e^x - 1)^3}$ to speed up your work.



Math 130: Lab XIII Answers

1. Carefully and quickly evaluate these polynomial and rational function limits at infinity by using dominant powers.

a) $\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to\infty} \frac{3x}{\sqrt{4x^2}} = \lim_{x\to\infty} \frac{3x}{|2x|} = \lim_{x\to\infty} \frac{3x}{2x} = \frac{3}{2}$ $\quad$ HA : $y = \frac{3}{2}$

b) $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+1}} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2}} = \lim_{x\to-\infty} \frac{3x}{|2x|} = \lim_{x\to-\infty} \frac{3x}{-2x} = -\frac{3}{2}$ $\quad$ HA : $y = -\frac{3}{2}$

c) $\lim_{x\to-\infty} \frac{|x|}{2x+1} \overset{\text{HP}}{=} \lim_{x\to-\infty} \frac{|x|}{2x} \overset{\text{x is negative}}{=} \lim_{x\to-\infty} \frac{-x}{2x} = -\frac{1}{2}$ $\quad$ HA : $y = -\frac{1}{2}$

d) $\frac{0}{0}$; $\lim_{x\to 0} \frac{\sin x^2}{2x} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{2x\cos x^2}{2} = \frac{0}{2} = 0$

e) $\frac{\infty}{\infty}$; $\lim_{x\to+\infty} \frac{\ln x^2}{x^2} \overset{\text{LH}}{=} \lim_{x\to+\infty} \frac{\frac{2x}{x^2}}{2x} = \lim_{x\to+\infty} \frac{1}{x^2} = 0$

f) $\frac{0}{0}$; $\lim_{x\to 0} \frac{5x-\sin 5x}{2x-\sin 2x} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{5-5\cos 5x}{2-2\cos 2x} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{25\sin 5x}{4\sin 2x} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{125\cos 5x}{8\cos 2x} = \frac{125}{8}$

2. Make sure to check those stages at which l'Hôpital's rule applies.

a) $0\cdot\infty$. $\lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}} \overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{2x^2}{x} = \lim_{x\to 0^+} -2x = 0.$

b) 0^0. Let $y = \lim_{x\to 0^+} (3x)^x$. We want to find y. Using the log process,

$$\begin{aligned}
\ln y = \ln(\lim_{x\to 0^+}(3x)^x) \Longrightarrow \ln y &\overset{\text{Cont}}{=} \lim_{x\to 0^+} \ln(3x)^x \\
\ln y &= \lim_{x\to 0^+} x\ln 3x \\
\ln y &\overset{\text{Recip}}{=} \lim_{x\to 0^+} \frac{\ln 3x}{\frac{1}{x}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{3}{3x}}{-\frac{1}{x^2}} \\
\ln y &= \lim_{x\to 0^+} -x = 0.
\end{aligned}$$

But $\ln y = 0$ implies $y = e^0 = 1$. So $\lim_{x\to 0^+} (3x)^x = y = 1$.

c) 1^∞. Let $y = \lim_{x\to 0^+} (1+2x)^{1/x}$, so

$$\begin{aligned}
\ln y &= \ln \lim_{x\to 0^+} (1+2x)^{1/x} \\
\ln y &\overset{\text{Cont}}{=} \lim_{x\to 0^+} \ln(1+2x)^{1/x} \\
\ln y &= \lim_{x\to 0^+} \frac{1}{x}\ln(1+2x) \\
\ln y &= \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to 0^+} \frac{\frac{2}{1+2x}}{1} = \frac{2}{1} = 2.
\end{aligned}$$

But $\ln y = 2$ implies $y = e^2$. So $\lim_{x\to 0^+} (1+2x)^{1/x} = e^2$.

d) 1^∞. Let $y = \lim_{x\to 0^+} (1+kx)^{1/x}$, so

$$\begin{aligned}
\ln y &= \ln \lim_{x\to+\infty} \left(1+\frac{k}{x}\right)^x \\
\ln y &\overset{\text{Cont}}{=} \lim_{x\to+\infty} \ln\left(1+\frac{k}{x}\right)^x
\end{aligned}$$

$$\begin{aligned}
\ln y &= \lim_{x\to+\infty} x\ln(1+\frac{k}{x}) \\
\ln y &= \lim_{x\to+\infty} \frac{\ln(1+\frac{k}{x})}{\frac{1}{x}} \\
\ln y &\overset{\text{l'Ho}}{=} \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{k}{x}}\cdot(-\frac{k}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{k}{1+\frac{k}{x}} = \frac{k}{1} = k.
\end{aligned}$$

But $\ln y = k$ implies $y = e^k$. So $\lim_{x\to+\infty}\left(1+\frac{k}{x}\right)^x = e^k$.

e) $\frac{0}{0}$. $\lim_{x\to 0}\frac{\arctan 4x}{\sin 2x} \overset{\text{l'Ho}}{=} \lim_{x\to 0}\frac{\frac{4}{1+16x^2}}{2\cos 2x} = \frac{\frac{4}{1}}{2} = 2.$

f) Not indeterminate. $\lim_{x\to 0}\frac{\sin 4x}{3\sec x} = \frac{0}{3} = 0.$ l'Hôpital's rule does not apply.

g) Not indeterminate. $\lim_{x\to 1^+}\frac{\overbrace{x^2+1}^{2}}{\underbrace{1-x}_{0^-}} = -\infty.$ l'Hôpital's rule does not apply.

h) Not indeterminate. $\lim_{x\to 0^+}\frac{\overbrace{\cos x}^{1}}{\underbrace{x^2}_{0^+}} = +\infty.$ l'Hôpital's rule does not apply.

i) $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{x\ln x}{x^2+1} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{1+\ln x}{2x} \overset{\text{l'Ho}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{2} = \frac{0}{2} = 0.$

3. Domain $(-\infty,\infty)$, all x. So there are no VAs.

HA: $\lim_{x\to+\infty}\frac{x}{x^2+1} = \lim_{x\to+\infty}\frac{1/x}{2+1/x^2} = 0$ and $\lim_{x\to-\infty}\lim_{x\to+\infty}\frac{x}{x^2+1} = \lim_{x\to+\infty}\frac{1/x}{2+1/x^2} = 0$. HA: $y=0$.

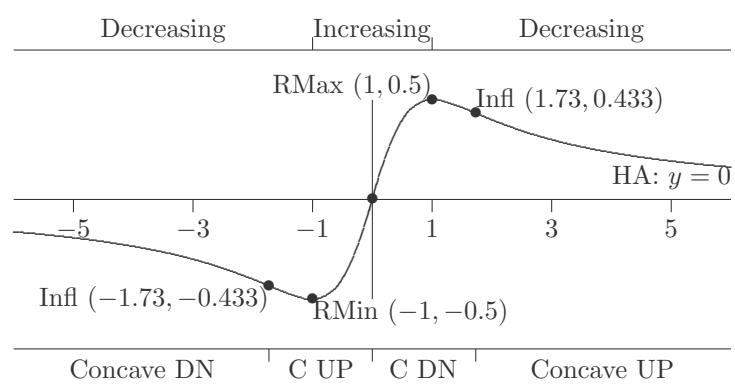
$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0$ at $x = \pm 1$.

$f''(x) = \frac{-2x(x^2+1)^2-(1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{-2x(x^2+1)-(1-x^2)(2)(2x)}{(x^2+1)^3} = \frac{2x^3-6x}{(x^2+1)^3} = 0$ at

Evaluate f. $f(0)=0$, $f(-1)=-\frac{1}{2}$, $f(1)=\frac{1}{2}$, $f(-\sqrt{3}) = -\frac{\sqrt{3}}{4} \approx -0.433$, and $f(\sqrt{3}) = \frac{\sqrt{3}}{4} \approx 0.433$.

		RMin		RMax	
f'	$--$	0	$++$	0	$--$
	Dec	-1	Inc	1	Dec

		Inf		Inf		Inf	
f''	$--$	0	$++$	0	$--$	0	$++$
	CD	$-(3^{1/2})$	CU	0	CD	$3^{1/2}$	CU



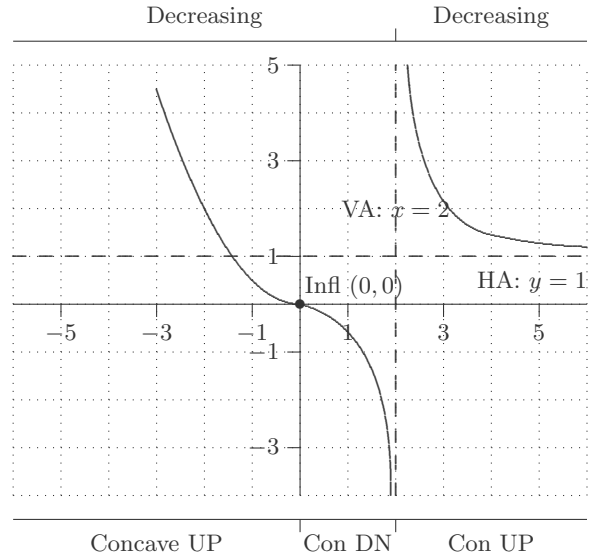
4. VA: $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 2^-} f(x) = -\infty$ so $x=2$ is a VA.

HA: $\lim_{x\to+\infty} f(x) = 1$, so $y=1$ is a HA. Also $\lim_{x\to-\infty} f(x) = +\infty$; no other HA.

Points: $f(0) = 0$

f'	$----$	0	$--$	NID	$--$		
	Dec	0	Dec	2	Dec		

		Inf					
f''	$+++$	0	$--$	NID	$++$		
	CU	0	CD	2	CU		



5. Domain all $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$.

VA: $\lim\limits_{x\to 0^+} \dfrac{\overbrace{2(x+1)^2}^{2}}{\underbrace{x^2}_{0^+}} = +\infty$ and $\lim\limits_{x\to 0^-} \dfrac{\overbrace{2(x+1)^2}^{2}}{\underbrace{x^2}_{0^+}} = +\infty$; so VA at $x = 0$.

HA: $\lim\limits_{x\to +\infty} \dfrac{2(x+1)^2}{x^2} = \lim\limits_{x\to +\infty} \dfrac{2x^2+4x+2}{x^2} = \lim\limits_{x\to +\infty} \dfrac{4+4/x+2/x^2}{2} = 2$, so HA at $y = 2$.

HA: $\lim\limits_{x\to -\infty} \dfrac{2(x+1)^2}{x^2} = \lim\limits_{x\to -\infty} \dfrac{2x^2+4x+2}{x^2} = \lim\limits_{x\to -\infty} \dfrac{4+4/x+2/x^2}{2} = 2$ so HA at $y = 2$ again.

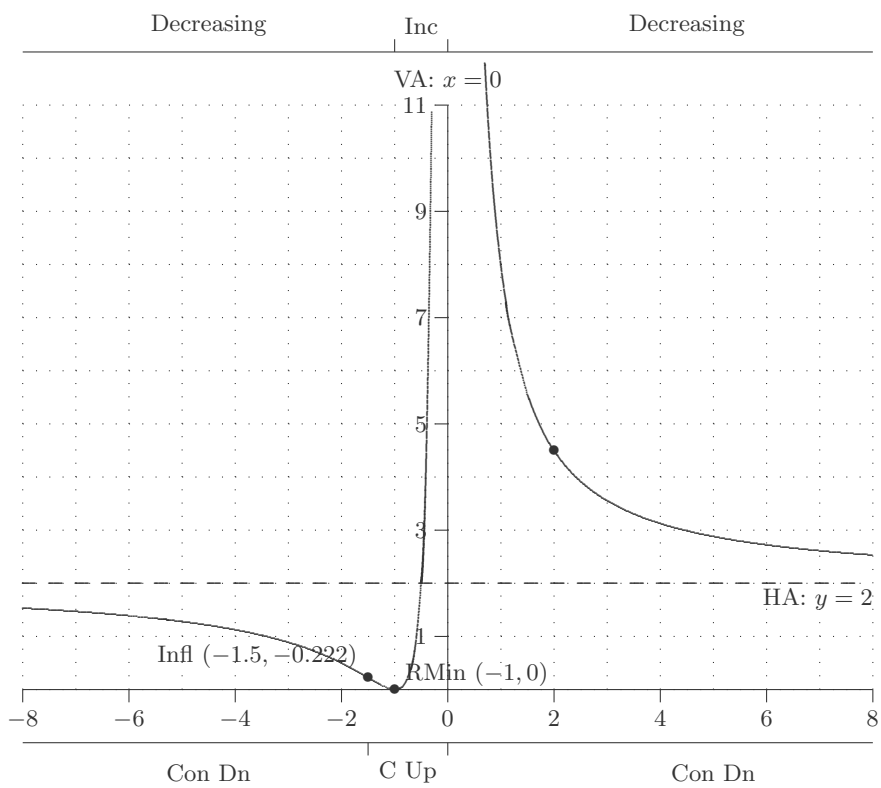
$f'(x) = \dfrac{-4x-4}{x^3} = 0$ at $x = -1$; $x = 0$ not in domain (NID).

$f''(x) = \dfrac{8x+12}{x^4} = 0$ at $x = -1.5$; $x = 0$ NID.

Evaluate f at key points. $f(-1) = 0$, $f(-1.5) = -0.\overline{22}$, and one other point $f(2) = 4.5$.

		RMin					
f'	$--$	0	$+$	NID	$---$		
	Dec	-1	Inc	0	Dec		

		Inf					
f''	$--$	0	$++$	NID	$+++$		
	CD	-1.5	CU	0.0	CU		



6. Bonus. Let $f(x) = \dfrac{1}{e^x - 1}$. Graph f.

Domain. Avoid $e^x - 1 = 0$, or $e^x = 1$ (when $x = 0$). The domain is: $(-\infty, 0) \cup (0, \infty)$.

VA:$\lim_{x\to 0^+} \dfrac{1}{\underbrace{e^x - 1}_{1^+ - 1 = 0^+}} = +\infty$. VA at $x = 0$.

VA: $\lim_{x\to 0^-} \dfrac{1}{\underbrace{e^x - 1}_{1^- - 1 = 0^-}} = -\infty$. VA at $x = 0$.

HA: $\displaystyle\lim_{x\to +\infty} \frac{1}{\underbrace{e^x - 1}_{+\infty}} = 0$, so HA at $y = 0$.

HA: $\displaystyle\lim_{x\to -\infty} \frac{1}{\underbrace{e^x - 1}_{0-1=-1}} = -1$, so HA at $y = -1$. Wow: Two different HA's. Cool!

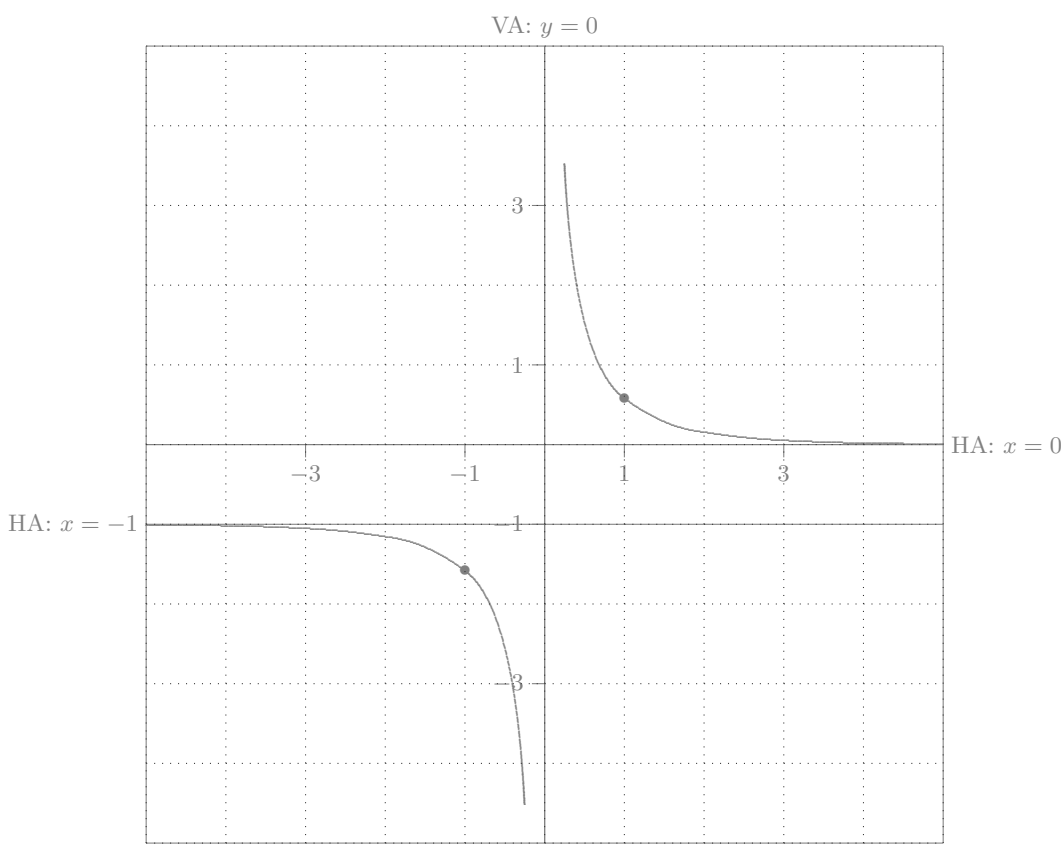
$f'(x) = \dfrac{0 - (1)e^x}{(e^x - 1)^2} = \dfrac{-e^x}{(e^x - 1)^2} \neq 0$. NID at $x = 0$.

f'	$----$	NID $*$	$----$
	Dec	0	Dec

Given $f''(x) = \dfrac{e^{2x} + e^x}{(e^x - 1)^3} \neq 0$. NID at $x = 0$.

f''	$----$	NID $*$	$++++$
	Con Dn	0	Con Up

Key points. No CPs. No POIs. No y-intercept (since $x = 0$ NID).

x-intercept ($y = 0$): $f(x) = \frac{1}{e^x - 1} \neq 0$. So no x-intercept.

Choose a point on either side of the VA at $x = 0$. Say $x = \pm 1$. Plot $f(-1) = -\frac{1}{e^{-1} - 1} \approx -1.582$, $f(1) \approx 0.582$.