Math 130: Lab XIV

1. These should be quick. Do in 10–15 minutes. Determine these indefinite integrals:

a) $\displaystyle\int cx^6 + de^x - be^{2x}\,dx = \frac{cx^7}{7} + de^x - \frac{be^{2x}}{2} + k$

b) $\displaystyle\int c\sqrt{x} + ax^{-4/3} - d\cos(4x)\,dx = \frac{2c}{3}x^{3/2} - 3ax^{-1/3} - \frac{d}{4}\sin(4x) + k$

c) $\displaystyle\int -\frac{8}{x} + \sin(4x) - 6 + 4\sec^2\frac{x}{6}\,dx = -8\ln(|x|) - \frac{\cos(4x)}{4} - 6x + 24\tan(x/6) + k$

d) $\displaystyle\int \frac{1}{8x^3} + 6^x \ln 6\,dx = -\frac{1}{16x^2} + 6^x + k$

e) $\displaystyle\int \frac{d}{\sqrt[b]{x^{b-1}}}\,dx = bdx^{1/b} + k$

f) $\displaystyle\int \frac{\sec(cx)\tan(cx)}{d}\,dx = \frac{\sec(cx)}{cd} + k$

g) $\displaystyle\int \frac{a^2}{\sqrt{bs - x^2}}\,dx = a^2\arcsin(x/b) + k$

h) $\displaystyle\int \frac{c}{a^2 + x^2}\,dx = \frac{c}{a}\arctan(x/a) + k$

2. Take another 10 to 15 minutes to determine these indefinite integrals:

a) $\displaystyle\int e^{cx} + 2\sin\left(\frac{x}{b}\right)\,dx = \frac{e^{cx}}{c} - 2b\cos(x/b) + k$

b) $\displaystyle\int d\cos(cx) - \sec^2(cx)\,dx = \frac{d}{c}\sin(cx) - \frac{\tan(cx)}{c} + k$

c) $\displaystyle\int -\frac{\cos x}{b} + a\sqrt[c-1]{x^c}\,dx = -\frac{\sin(x)}{b} + \frac{a(c-1)}{2c-1}x^{(2c-1)/(c-1)} + k$

d) $\displaystyle\int 4x^{-7/5} - \frac{1}{\sqrt[4]{x^7}} + \,dx = \frac{4}{3}x^{-3/4} - 10x^{-2/5} + k$

e) $\displaystyle\int \frac{2x^3 - a\sqrt{x} + b}{x}\,dx = \int 2x^2 - ax^{-1/2} + \frac{b}{x}\,dx = \frac{2}{3}x^3 - 2ax^{1/2} + b\ln|x| + k$

f) $\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^x}\,dx = \int e^x - e^{-3x}\,dx = e^x + \frac{e^{-3x}}{3} + k$. Hint: Simplify first.

g) $\displaystyle\int a\sqrt{x}(6x^3 - 2x)\,dx = \int 6ax^{7/2} - 12ax^{3/2}\,dx = \frac{2a}{9}x^{9/2} - \frac{4a}{5}x^{5/2} + k$

3. Take 5 minutes:

a) Consider the function $f(t) = 6t^2 + 4t + 2$. Let $F(t)$ be the antiderivative of $f(t)$ with $F(1) = 4$. Find $F(t)$.

b) Consider the function $f(t) = e^t + 2t$. Let $F(t)$ be the antiderivative of $f(t)$ with $F(1) = 2$. Find $F(t)$.

Solution.

a) The antiderivative of $f'(x) = 6x^2 + 4x + 2$ is $f(x) = 2x^3 + 2x^2 + +2x + c$. But then $f(1) = 2 + 2 + 2 + c = 4$. So $c = -2$ and $f(x) = 2x^3 + 2x^2 + 2x - 2$.

b) The antiderivative of $f'(t) = e^t + 2t$ is $f(t) = e^t + t^2 + c$. But $f(1) = 2 = e + 1 + c$. So $c = 1 - e$. And $f(t) = e^t + t^2 + 1 - e$.

4. Take 5 minutes:

a) Given $f''(x) = 4x + 5$ with $f'(1) = 0$ and $f(0) = -4$. Find $f'(x)$.

b) Now find $f(2)$. Hint: First find $f(x)$.

Solution.

a) The antiderivative of $f''(x) = 4x + 5$ is $f'(x) = 2x^2 + 5x + c$. But $f'(1) = 2 + 5 + c = 0$. So $c = -7$ and $f'(x) = 2x^2 + 5x - 7$.

b) The antiderivative of $f'(x) = 2x^2 + 5x$ is $f(x) = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x + c$. But $f(0) = 0 + 0 - 0 + c = -4$. So $c = -4$ and $f(x) = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 7x - 4$. So $f(2) = \frac{16}{3} + 10 - 14 - 4 = -\frac{8}{3}$.

5. Take 10 minutes: A stone is thrown straight upward from the edge of a cliff, 80 feet above a river, at a speed of 64 feet per second. Remember that the acceleration due to gravity is -32 ft/s^2.

a) The velocity of the object at time t is $v(t) =$

b) The position of the object at time t is $s(t) =$

c) Determine the time when the stone reaches its highest point ____________ and the height at this time is ____________

d) At what time does the stone hit the ground?

e) What is the velocity of the stone when it hits the ground?

Solution.

a) $v(t) = \int -32\, dt = -32t + c$. $v(0) = 0 + c = 64$, so $c = 64$. So $v(t) = -32t + 64$.

b) $s(t) = \int -32t + 64\, dt = -16t^2 + 64t + c$. $s(0) = 0 + 0 + c = 80$, so $c = 80$. So $s(t) = -16t^2 + 64t + 80$.

c) Highest point when velocity is 0: $v(t) = -32t + 64 = 0$, so $t = 2$s. Height: $s(2) = -64 + 128 + 80 = 144$ft.

d) Hits ground when height is 0: $s(t) = -16t^2 + 64t + 80 = -16(t^2 - 4t - 5) = -16(t-5)(t+1) = 0$ at $t = 5$ ($t \neq -1$).

e) The velocity of the stone when it hits the ground is $v(5) = -32(5) + 64 = -96$ ft/s.

6. Take 10 minutes: Mo Green is attempting to run the 100m dash in the Geneva Invitational Track Meet in 9.8 seconds. He wants to run in a way that his *acceleration* is constant, a, over the entire race.

a) Determine his velocity function, $v(t)$ (a will still appear as an unknown constant.)

b) Determine his position function, $s(t) =$ There should be no unknown constants in your function. Use exact values (fractions).

c) What is his velocity at the end of the race? Use exact values (fractions). Do you think this is realistic?

Solution.

a) We have: constant acceleration $= a\,\text{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

b) And

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2}\,\text{m/s}^2$. So $s(t) = \frac{1}{2}at^2 = \frac{100}{(9.8)^2}t^2$.

c) Mo's velocity at the end of the race is $v(9.8) = a(9.8) = \frac{200}{(9.8)^2} \cdot 9.8 = \frac{200}{9.8} \approx 20.4$ m/s... not realistic.

7. France has been in the vanguard of high-speed passenger rail travel since the 1970s, and now has a modern rail network capable of accommodating trains running at speeds in excess of 84 m/s (about 300 km/h). Suppose such a train is approaching a station at 84 m/s and begins braking (say at time $t = 0$) at a constant rate of 2.8 m/s^2. How far (in meters) from the railway station did it begin to brake if it stopped right at the station platform? Use the steps below.

a) First determine the velocity function $v(t)$

b) Next, determine how long it takes the train to stop.

c) How far (in meters) from the railway station did it begin to brake if it stopped right at the station platform? (You will need the position function.)

Solution.

a) We have: constant acceleration $= -2.\,\text{m/s}^2$; $v_0 = 84$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = -2.8t + 84.$$

b) The train stops when $v(t) = -2.8t + 84 = 0$. Solving gives $t = 84/1.4 = 30$ s

c) And

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -1.4t^2 + 84t.$$

So the train travels $s(30) = -1.4(30)^2 + 84(30) = 1260$ m.

8. a) A person drops a stone from a bridge. What is the height (in meters) of the bridge if the person hears the splash 5 seconds after dropping it? [Remember: Acceleration due to gravity in -9.8 m/s^2.]

Solution.

a) We have: constant acceleration $= a\,\text{m/s}^2$; $v_0 = 0$ m/s; $s_0 = 0$ m. So

$$v(t) = at + v_0 = at$$

b) And

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = \tfrac{1}{2}at^2.$$

But $s(9.8) = \frac{1}{2}a(9.8)^2 = 100$, so $a = \frac{200}{(9.8)^2}$ m/s^2. So $s(t) = \frac{1}{2}at^2 = \frac{100}{(9.8)^2}t^2$.

c) Mo's velocity at the end of the race is $v(9.8) = a(9.8) = \frac{200}{(9.8)^2} \cdot 9.8 = \frac{200}{9.8}$ m/s. . . not realistic.

d) **Extra Credit**: [Hand in by Friday] Did you take into account that sound does not travel instantaneously in your calculation above? Assume that sound travels at 340.3 m/s. What is the height (in m) of the bridge if the person hears the splash 5 seconds after dropping it?

Solution.

a) Here's what we know. $v_0 = 0$ (dropped) and $s(5) = 0$ (hits water). And we know acceleration is constant, $a = -9.8$ m/s^2. We want to find the height of the bridge, which is just s_0. Use our constant acceleration motion formulas to solve for a.

$$v(t) = at + v_0 = -9.8t$$

and

$$s(t) = \tfrac{1}{2}at^2 + v_0t + s_0 = -4.9t^2 + s_0.$$

Now we use the position we know: $s(5) = 0$.

$$s(5) = -4.9(5)^2 + s_0 \Rightarrow s_0 = 122.5.$$

Notice that we did not need to use the velocity function.

b) Let s_0 denote the initial position (height). Let t^* denote the time it takes the stone to reach the water. Since you hear it splash 5 seconds after you drop it, then the time it takes the sound to travel back to you is $5 - t^*$ seconds. Since the sound travels at a constant velocity of 340.3 m/s, the distance the sound travels is $340.3(5-t^*)$ m (upward). So $s_0 = 340.3(5 - t^*)$ m. (since the stone travels the same distance, but downwards). As in the first part of the problem, $s(t) = -4.9t^2 + s_0$. Since t^* is the time it takes the stone to reach the water, we know that $s(t^*) = 0$ So using the value of s_0

$$s(t^*) = -4.9(t^*)^2 + s_0 = -4.9(t^*)^2 + 340.3(5 - t^*) = 0.$$

So

$$-4.9(t^*)^2 - 340.3t^* + 1701.5 = 0.$$

Use the quadratic formula to find

$$t^* = \frac{340.3 \pm \sqrt{(-340.3)^2 - 4(-4.9 \cdot 1701.5)}}{2(-4.9)} = 4.684076\,\text{s}$$

So, from above

$$s_0 = 340.3(5 - t^*) = 340.3(5 - 4.684076) = 107.5089\,\text{m}.$$