Math 130: NOT Test 2. With Answers. Most Recent Update: 2016/12/09 at 07:33:18

Office Hours (LN 301/301.5): M 3:30-4:30, Tu 11:00-1:00, W 12:15-1:15, F 1:30-2:30. Other times by appointment. **Math Intern:** Sun through Thurs: 3:00–6:00, 7:00–10:00pm. **Website:** Use the links at the course homepage on **Canvas** or go to my course Webpage: `http://math.hws.edu/~mitchell/Math130F16/index.html`.

Derivative Practice

1. Here are some fairly simple functions. Find the derivatives of each.

a) $2x^{2/3}$ **b)** $\cos x$ **c)** $\tan 3x$ **d)** $\dfrac{\sqrt[3]{x^5}}{2}$ **e)** $\dfrac{3}{2x^2}$

f) $\sin(-2x)$ **g)** $e^{\frac{x}{4}}$ **h)** $\dfrac{\sec(\pi x)}{5}$ **i)** $\sin(2x+1)$ **j)** e^{4x+2}

k) Find the equation (not just the slope) of the **tangent line** to the function in part (a) at $x = 1$.

2. Ok, here are some exponential and log functions. Take their derivatives.

a) $2e^x \sin x$ **b)** $e^x + x^e + x + e$ **c)** $\dfrac{e^x - 1}{e^x + 1}$ **d)** $\ln(\ln(3x+1))$

e) $6x^4 e^x$ **f)** $\dfrac{\tan x}{e^x}$ **g)** $4e^x \sec x$

h) $\ln\left(\dfrac{e^x}{e^x+1}\right)$ log rules **i)** $2e^{-x}$ **j)** $5\ln x$

k) $\ln|5x|$ **l)** $\ln(5x+1)$ **m)** $e^x \ln|x|$ **n)** $\dfrac{\ln x}{\ln(x+1)}$

3. Now try these problems that combine derivative rules. Use log rules where helpful.

a) $2e^x \tan x$ **b)** $\dfrac{e^{2x}}{2+\sin x}$ **c)** $(1 + x^2 \tan x)^5$ **d)** $\sqrt[3]{x^4 + 2x^5}$

e) $\cos(e^{2x^3+2x})$ **f)** $e^{\cos(2x^3+2x)}$ **g)** $\ln\sqrt{2x^3+1}$ **h)** $\sqrt{\ln(2x^3+1)}$

4. Here are some harder derivative problems to try.

a) $\sec(3x^2 + x)$ **b)** e^{3x^2+x} **c)** $\sqrt[4]{3x^2+x}$

d) $\left(\dfrac{x^2-1}{x^2+1}\right)^4$ **e)** $\cos(3x)\cos(2x)$ **f)** $((x^3+2x)^4+5)^6$

g) $\sin(\cos(\tan 2x))$ **h)** $\dfrac{2}{(t^2-5t+4)^4}$ **i)** $\tan(x^2) + \tan^2(x) + \tan^2(x^2)$

j) $\sec(x^2 e^{10x})$ **k)** $\ln\left|\dfrac{e^{4x}-1}{x^2+11}\right|$ **l)** $\dfrac{\ln(t)}{t^3}$ **m)** $\ln\sqrt[3]{9x^2+11}$

5. Find the derivatives of these functions using the derivative formula for a general exponential function that we developed.

a) $5 \cdot 6^x$ **b)** $x^4 \cdot 4^x$ **c)** $3^{x^2+\tan x}$ **d)** $4^{[x^2+1]^3}$

e) For which values of x does $x^4 \cdot 4^x$ have a horizontal tangent?

6. Use Logarithmic Differentiation to find the derivatives of the following functions. (I probably would not tell you that on the test. How can you tell that you should use this technique?)

a) $y = x^{\sec x}$ **b)** $y = (\sec x)^x$ **c)** $y = x^{\ln x}$ **d)** $y = (\ln x)^x$ **e)** $y = (x^2+1)^{x^2+1}$

f) Compare parts (c) and (d) to the derivative of $y = \ln(x^x)$

Derivative Theory

7. **a)** Carefully state the **limit definition of** $f'(x)$.

b) Carefully use the **limit definition** to find the derivative of $f(x) = x^2 - 4x$.

c) Carefully use the **limit definition** to find the derivative of $f(x) = \frac{1}{2x}$.

d) Carefully use the **limit definition** to find the derivative of $f(x) = \sqrt{5x}$.

e) In part (a) you should have said $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. So every time you see a limit of this form, it represents the derivative of some function. For example, (notice that $f(x)$ is easy to spot)

$$\lim_{h\to 0} \frac{\overbrace{\sec(x+h)}^{f(x+h)} - \overbrace{\sec(x)}^{f(x)}}{h} \overset{\text{definition}}{=} D_x[(\sec(x)] = \sec(x)\tan(x).$$

So what is $\lim_{h\to 0} \dfrac{e^{3(x+h)} - e^{3x}}{h}$? [Hint: First figure out what $f(x)$ is.]

f) What is $\lim_{h\to 0} \dfrac{\ln(x+h) - \ln x}{h}$?

g) We can use this idea to evaluate limits at points as well.

$$\lim_{h\to 0} \frac{\overbrace{\sin(0+h)}^{f(0+h)} - \overbrace{\sin(0)}^{f(0)}}{h} \overset{\text{definition}}{=} D_x[(\sin(x)]\Big|_{x=0} = \cos(x)\Big|_{x=0} = \cos(0) = 1.$$

So what is $\lim_{h\to 0} \dfrac{\ln(2+h) - \ln 2}{h}$? [Hint: First figure out what $f(x)$ is.]

h) Use the same idea to determine $\lim_{h\to 0} \dfrac{\cos(\frac{\pi}{2}+h) - \cos(\frac{\pi}{2})}{h}$

i) Draw a differentiable function that does NOT have an inverse. What can you say about the slope of such a function?

8. **a)** In the course of doing a complicated limit problem, a calculus student encountered the following situation: She knew that f and g were **differentiable** functions and she had to evaluate

$$\lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] = \underline{\hspace{6cm}}$$

Fill in the the limit (in terms of the functions f and g) and justify your answer using your knowledge of calculus **definitions** and **theorems** in no more than 2 short sentences.

b) Let $f(x)$ be a differentiable function. Use the **limit definition** of the derivative to find the derivative of the function $G(x) = 2xf(x)$. Note any key properties of f that you use. (Check your answer using a basic derivative rule.)

9. Define $f(x) = \begin{cases} \sin(2x), & \text{if } x > 0 \\ 2x^2 + x, & \text{if } x \le 0 \end{cases}$.

a) Evaluate $f(0)$. Which function definition must you use? NOTE: Every time you need to use $f(0)$ later in this problem, you must use the value you just calculated.

b) Is f continuous at $x = 0$? Explain carefully (using one-sided limits).

c) Is f differentiable at $x = 0$? Explain carefully (using one-sided limits with the definition of the derivative at a point). For example: The derivative from the right is

$$\text{Right Derivative}: \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$$

where you must substitute the appropriate function definitions for $f(x)$ and $f(0)$.

10. Review Lab 5, Problem 6 (http://math.hws.edu/~mitchell/Math130F16/Lab05-F-16.pdf) and its answer there.

11. On the exam, Professor Mitchell asks what is the derivative of $\cot x$. You did not study this because he said it would not be on the exam. You did study the derivatives of $\sin x$, $\cos x$, $\tan x$, and $\sec x$. Show how you can find the derivative of $\cot x$ by using these other derivatives and basic derivative rules.

12. Let $s(t) = te^t$ be the displacement (meters) or position of an object at time t (seconds).

a) Determine the velocity, acceleration, and the jerk. (Remember, acceleration is the derivative of velocity, and jerk is the derivative of acceleration). Simplify each before going to the next.

b) When is the object stationary?

c) What is the 100th derivative of $s(t)$? Hint: You should have noticed a pattern.

13. Let $f(x) = x\sin(\pi x)$. Find the equation of the tangent line at $x = 1$.

14. Find the derivatives of the following.

a) $y = \dfrac{\sqrt[3]{x^5}}{10} + \dfrac{x^4}{10} - \dfrac{2}{3x}$ **b)** $g(x) = 8 + x^3 \sec x$ **c)** $f(x) = e^{1+\sec(x^2)}$

d) $g(x) = \dfrac{x^2-1}{x^2+1}$ **e)** $h(t) = \sqrt[4]{5t^3+1}$ **f)** $g(x) = \cos^3 x - \sin^3 x$

g) $g(x) = (12 + xe^x)^4$ **h)** $f(x) = e^2 + e^x \tan(e^{3x})$ **i)** $h(x) = 6 - \dfrac{\sin(x)\cos(6x)}{3}$

j) $p(t) = te^{t^2+1}$ **k)** $\ln(\cos x)$ **l)** $f(t) = \ln(8t^2+1)$

m) $\dfrac{\ln(t)}{t^3}$ **n)** $f(x) = \ln\left|\dfrac{x^3 - x^2 + 4}{2x^3 - x}\right|$ **o)** $f(x) = \ln\sqrt[3]{9x^2+11}$ **p)** $r(t) = \dfrac{t^3 - t^2}{2t^3 - t}$

Implicit Differentiation and Chain Rule

15. Now that we know about **implicit differentiation**, we can use it to find the derivative of $\ln x$ (if we did not already know the answer) We can find the derivative of $y = \ln x$ because $\ln x$ has an inverse, namely e^x, and we know the derivative for the inverse. Here's the proof in step-by-step form: Fill in the blanks.

a) Let $y = \ln x$. We want to find $\frac{dy}{dx}$. Start with:

$$y = \ln x$$

Now apply the inverse of $\ln x$ to each side and simplify.

$$\underline{\qquad\qquad} = \underline{\qquad}$$

$$\underline{\qquad\qquad} = \underline{\qquad}$$

b) Now take the derivative of both sides using implicit differentiation on the left:

$$D_x[\qquad] = D_x[\qquad]$$

$$\underline{\qquad\qquad} = \underline{\qquad}$$

c) Solve for $\frac{dy}{dx}$. $\frac{dy}{dx} =$ _____. Then substitute for e^y using the second line you filled in in part (a). Did you get the correct formula for the derivative of $y = \ln x$?

16. **a)** Given the relation $x^3 - x^2y + y^2 = 13$, find $\frac{dy}{dx}$.

b) Find the tangent line to the curve at the point $(2, -1)$

c) Given the relation $x^2y^2 + e^{2y} = x$, find $\frac{dy}{dx}$.

17. This problem tests whether you understand the chain rule. Quickly compute:

a) $\dfrac{d}{dx}[4\sec(x)] =$ **b)** $\dfrac{d}{dx}[\sec(4x)] =$ **c)** $\dfrac{d}{dx}\left[\sec(\frac{x}{4})\right] =$

d) $\dfrac{d}{dx}\left[\dfrac{\sec x}{4}\right] =$ **e)** $\dfrac{d}{dx}[\sec(x^4)] =$ **f)** $\dfrac{d}{dx}[\sec^4 x] =$

18. For each relation, find $\frac{dy}{dx}$.

a) $3y + \ln y = x^2 + x$ **b)** $y^2 - x^2 y = 2$ **c)** $e^{xy^2} = x^2 - 4$ **d)** $x^2 + x^2y^2 + y^3 = 3$ **e)** $xy^2 = 18$

More Theory

19. True or false. If false, draw a function which illustrates your answer.

a) If f is differentiable at $x = a$, then f is continuous at $x = a$.

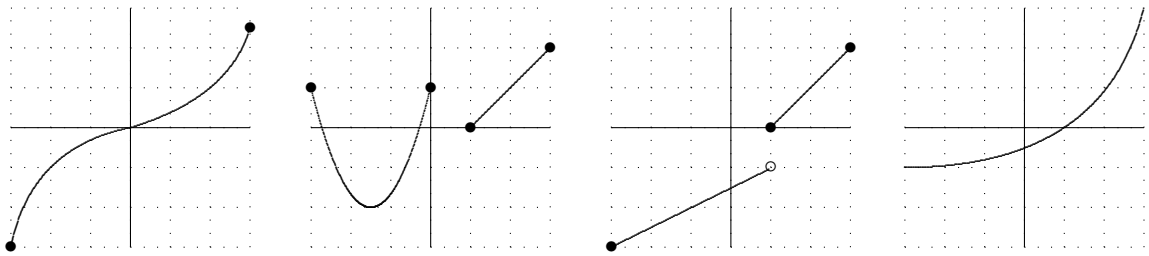
b) If f is continuous at $x = a$, then f is differentiable at $x = a$.

c) If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

20. **a)** Carefully state the definition of $g(x)$ being the **inverse of** $f(x)$.

b) Define what it means for f to be a **one-to-one** function.

21. Explain which of these functions have inverses and which do not. If a function has an inverse, graph it on the same set of axes.

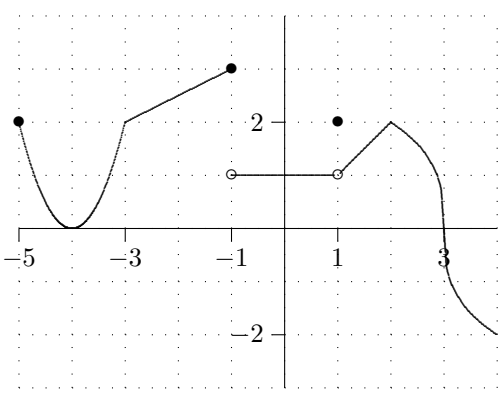


22. **a)** Fill in the table. Use the graph or the information given.

b) Find all points where f is *not continuous.*

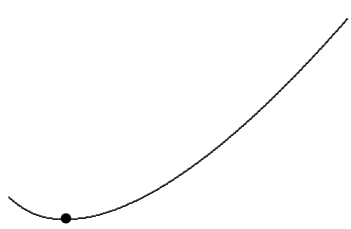
c) Find all points $x = a$ where f is *not differentiable* even though f is continuous at $x = a$.

d) Find a point $x = a$ where f is *not continuous* even though f is differentiable at $x = a$.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Ctns	Diff'ble
-4						
-3						
-2						
-1						
0						
1						
2						
3						
5	3					Yes
8			4		No	

23.



Someone in my previous calculus class handed in the graph to the left. It was labeled $f(x) = x \ln x$ but there were no axes or coordinates. Determine both the x and y coordinates of the low point of the graph marked with •. Hint: What is the slope of the tangent line there?

24. **a)** What is $\lim_{h\to 0} \dfrac{e^h - 1}{h}$? Hint: Use problem 7(g).

b) Use part (a) to determine $\lim_{h\to 0} \dfrac{e^{2h} - 1}{h}$. Hint: Factor the numerator.

25. Find the derivatives of these inverse trig functions:

a) $\frac{d}{dx}\arctan(6x^2)]$ **b)** $\frac{d}{dx}[\arcsin(\sqrt{x})]$ **c)** $\frac{d}{dx}[\arctan(e^{2x})]$ **d)** $\frac{d}{dx}[\arcsin(\arcsin x)]$

e) $\frac{d}{dx}[\arctan(\ln|6x|)]$ **f)** $\frac{d}{dx}[\arcsin(6e^{\sin x})]$ **g)** $\frac{d}{dx}(e^{2\arcsin x^2})$ **h)** $\frac{d}{dx}[(\arcsin 2x)(\tan 5x^2)]$

i) $\frac{d}{dx}(\ln|\arctan e^{x^4+1}|)$

Math 130: Answers to NOT Test 2

1. Answers:

a) $\dfrac{4x^{-1/3}}{3}$ **b)** $-\sin x$ **c)** $3\sec^2 3x$ **d)** $D_x(\frac{1}{2}x^{5/3}) = \frac{5x^{2/3}}{6}$ **e)** $D_x(\frac{3}{2}x^{-2}) = -3x^{-3}$

f) $-2\cos(-2x)$ **g)** $\dfrac{e^{\frac{x}{4}}}{4}$ **h)** $\dfrac{\pi\sec(\pi x)\tan(\pi x)}{5}$ **i)** $2\cos(2x+1)$ **j)** $4e^{4x+2}$

k) Slope: $f'(1) = \frac{4}{3}$. Point: $(1, f(1)) = (1, 2)$. Therefore the equation is $y - 2 = \frac{4}{3}(x-1)$ or $y = \frac{4}{3}x + \frac{2}{3}$.

2. Answers:

a) $2e^x(\sin x + \cos x)$ **b)** $e^x + ex^{e-1} + 1$ **c)** $\dfrac{2e^x}{(e^x+1)^2}$ **d)** $\dfrac{1}{\ln(3x+1)} \cdot \dfrac{3}{3x+1}$

e) $(6x^4 + 24x^3)e^x$ **f)** $\dfrac{\sec^2 x - \tan x}{e^x}$ **g)** $4e^x \sec x(1 + \tan x)$

h) $1 - \dfrac{e^x}{e^x+1}$ **i)** $-2e^{-x}$ **j)** $\dfrac{5}{x}$ **k)** $\dfrac{5}{5x} = \dfrac{1}{x}$

l) $\dfrac{5}{5x+1}$ **m)** $e^x \ln|x| + \dfrac{e^x}{x}$ **n)** $\dfrac{\frac{\ln(x+1)}{x} - \frac{\ln x}{x+1}}{[\ln(x+1)]^2}$

3. Answers:

a) $2e^x(\tan x + \sec^2 x)$ **b)** $\dfrac{e^{2x}(4 + 2\sin x - \cos x)}{(2+\sin x)^2}$

c) $5(1 + x^2\tan x)^4(2x\tan x + x^2\sec^2 x)$ **d)** $\dfrac{1}{3}(x^4 + 2x^5)^{-2/3}(4x^3 + 10x^4)$

e) $-(6x^2+2)e^{2x^3+2x}\sin(e^{2x^3+2x})$ **f)** $-(6x^2+2)\sin(2x^3+2x) \cdot e^{\cos(2x^3+2x)}$

g) $D_x\left[\frac{1}{2}\ln(2x^3+1)\right] = \frac{1}{2} \cdot \dfrac{6x^2}{2x^3+1} = \dfrac{3x^2}{2x^3+1}$ **h)** $\frac{1}{2}\left[\ln(2x^3+1)\right]^{-1/2} \cdot \dfrac{6x^2}{2x^3+1} = \dfrac{3x^2}{(2x^3+1)[\ln(2x^3+1)]^{1/2}}$

4. Answers:

a) $(6x+1)\sec(3x^2+x)\tan(3x^2+x)$ **b)** $(6x+1)e^{3x^2+x}$

c) $\dfrac{6x+1}{4(3x^2+x)^{3/4}}$ **d)** $4\left(\dfrac{x^2-1}{x^2+1}\right)^3 \cdot \dfrac{x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} = 4\left(\dfrac{x^2-1}{x^2+1}\right)^3 \cdot \dfrac{4x}{(x^2+1)^2} = \dfrac{16x(x^2-1)^3}{(x^2+1)^5}$

e) $-3\sin(3x)\cos(2x) - 2\cos(3x)\sin(2x)$ **f)** $24(3x^2+2)(x^3+2x)^3((x^3+2x)^4+5)^5$

g) $-2\cos(\cos(\tan 2x))(\sin(\tan 2x))\sec^2 2x$ **h)** $= D_t[2(t^2-5t+4)^{-4}] = -8(2t-5)(t^2-5t+4)^{-5}$

i) $2x\sec^2(x^2) + 2\tan(x)\sec^2(x) + 4x\tan(x^2)\sec^2(x^2)$

j) $(2xe^{10x} + 10x^2e^{10x})\sec(x^2e^{10x})\tan(x^2e^{10x})$ **k)** $D_x[\ln|e^{4x}-1| - \ln|x^2+11|] = \dfrac{4e^{4x}}{e^{4x}-1} - \dfrac{2x}{x^2+11}$

l) $\dfrac{\frac{1}{t}\cdot t^3 - \ln t \cdot 3t^2}{(t^3)^2} = \dfrac{1 - 3\ln t}{t^4}$ **m)** $D_t\left[\dfrac{1}{3}\cdot\ln 9x^2 + 11\right] = \dfrac{1}{3}\cdot\dfrac{1}{9x^2+11}\cdot(18x) = \dfrac{6x}{9x^2+11}$

5. a) $\dfrac{d}{dx}[5\cdot 6^x] = 5\cdot 6^x\ln 6 = 5\ln 6(6^x)$.

b) $\dfrac{d}{dx}[x^4\cdot 4^x] = 4x^3\cdot 4^x + x^4\cdot 4^x\ln 4 = x^3\cdot 4^x[4 + x\ln 4]$.

c) $y' = 3^{x^2+\tan x}\cdot\ln 3\cdot(2x + \sec^2 x)$.

d) $y' = 6x[x^2+1]^2 4^{[x^2+1]^3}\ln 4$

e) From part (b), the slope is 0 when $x^3\cdot 4^x[4 + x\ln 4] = 0$. Therefore $x = 0$ or $x = -\dfrac{4}{\ln 4}$.

6. Use Logarithmic Differentiation.

a) $\ln y = \ln(x^{\sec x}) = \sec x \cdot \ln x$. So $\dfrac{1}{y}\dfrac{dy}{dx} = \sec x \tan x \ln x + \dfrac{\sec x}{x}$ so $\dfrac{dy}{dx} = x^{\sec x}\left(\sec x \tan x \ln x + \dfrac{\sec x}{x}\right)$.

b) $\ln y = \ln(\sec x)^x = x\ln(\sec x)$. So $\dfrac{1}{y}\dfrac{dy}{dx} = \ln(\sec x) + x\tan x$ so $\dfrac{dy}{dx} = (\sec x)^x\,(\ln(\sec x) + x\tan x)$.

c) $\ln y = \ln(x^{\ln x}) = \ln x \ln x = (\ln x)^2$. So $\dfrac{1}{y}\dfrac{dy}{dx} = 2\ln x \cdot \dfrac{1}{x}$. So $\dfrac{dy}{dx} = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$.

d) $\ln y = \ln(\ln x)^x = x\ln(\ln x)$. So $\dfrac{1}{y}\dfrac{dy}{dx} = \ln(\ln(x)) + \dfrac{x}{\ln x}\cdot\dfrac{1}{x}$. So $\dfrac{dy}{dx} = (\ln x)^x\left(\ln(\ln(x)) + \dfrac{1}{\ln x}\right)$.

e) $\ln y = \ln(x^2+1)^{x^2+1}) = (x^2+1)\ln(x^2+1)$. So $\dfrac{1}{y}\dfrac{dy}{dx} = 2x\ln(x^2+1) + (x^2+1)\cdot\dfrac{1}{x^2+1}\cdot 2x$. So $\dfrac{dy}{dx} = (x^2+1)^{x^2+1}\left(2x\ln(x^2+1) + 2x\right)$.

f) $D_x(\ln x^x) = D_x(x\ln x) = \ln x + x\cdot\frac{1}{x} = \ln x + 1$.

7. a) $f(x)$ is differentiable if $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists. Notice that each of the parts below starts with this definition.

b) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - 4(x+h) - [x^2-4x]}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2-4h}{h} = \lim_{h\to 0} 2x+h-4 = 2x-4.$$

c) $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{2(x+h)}-\frac{1}{2x}}{h} = \lim_{h\to 0}\frac{\frac{2x-2(x+h)}{2(x+h)(2x)}}{h} = \lim_{h\to 0}\frac{-2h}{2(x+h)(2x)h} = \lim_{h\to 0}\frac{-1}{(x+h)(2x)} = \frac{-1}{2x^2}$$

d) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{5(x+h)}-\sqrt{5x}}{h}\cdot\frac{\sqrt{5(x+h)}+\sqrt{5x}}{\sqrt{5(x+h)}+\sqrt{5x}}$$

$$= \lim_{h\to 0}\frac{5(x+h)-5x}{h\cdot[\sqrt{5(x+h)}+\sqrt{5x}]} = \lim_{h\to 0}\frac{5x+5h-5x}{h\cdot[\sqrt{5(x+h)}+\sqrt{5x}]} = \lim_{h\to 0}\frac{5h}{h\cdot[\sqrt{5(x+h)}+\sqrt{5x}]}$$

$$= \lim_{h\to 0}\frac{5}{\sqrt{5(x+h)}+\sqrt{5x}} = \frac{5}{\sqrt{5x}+\sqrt{5x}} = \frac{5}{2\sqrt{5x}.}$$

e) Here $f(x) = e^{3x}$; the limit is just $f'(x) = 3e^{3x}$.

f) This time $f(x) = \ln x$; the limit is just $f'(x) = \frac{1}{x}$.

g) $$\lim_{h\to 0}\frac{\ln(2+h)-\ln 2}{h} = D_x[(\ln(x)]\Big|_{x=2} = \frac{1}{x}\Big|_{x=2} = \frac{1}{2}.$$

h) $$\lim_{h\to 0}\frac{\cos(\frac{\pi}{2}+h)-\cos(\frac{\pi}{2})}{h} = D_x[(\cos(x)]\Big|_{x=\pi/2} = -\sin(x)\Big|_{x=\pi/2} = -\sin(\pi/2) = -1.$$

i) `Your graph should be connected (continuous) and smooth (no corners, differentiable) AND it must fail the horizontal line test. Notice that the function must either go up and then down or down and then up to fail the horizontal line test. So the the slope of` f `is positive at some points and negative at others (and likely 0 in between).`

8. a) We are told f is differentiable, so it is continuous, therefore $\lim_{h\to 0} f(x+h) = f(x)$—we can just 'plug in' to evaluate the limit. *You need to say these things.* They are like "magic words" so that I know that you know how the limit is being calculated. Further g is differentiable so $\lim_{h\to 0}\frac{g(x+h)-g(h)}{h}$ is just the definition of $g'(x)$. We have

$$\lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(0)}{h}\right] = \underline{f(x)g'(x).}$$

b) $$G'(x) = \lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to 0}\frac{2(x+h)f(x+h)-2xf(x)}{h}$$

$$= \lim_{h\to 0}\frac{2xf(x+h)+2hf(x+h)-2xf(x)}{h} = \lim_{h\to 0}\frac{2x[f(x+h)-f(x)]+2hf(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{2x[f(x+h)-f(x)]}{h} + \frac{2hf(x+h)}{h} = \lim_{h\to 0} \frac{2x[f(x+h)-f(x)]}{h} + 2f(x+h) = 2xf'(x) + 2f(x).$$

Notice to evaluate the limit at the last step, we have used the definition of the derivative and the fact that $\lim_{h\to 0} f(x+h) = f(x)$ because f is continuous—we can just plug $h = 0$ into the function—and f is continuous because it is differentiable. You need to say these things. They are like "magic words" so that I know that you know how the limit is being calculated.

9. a) Use the definition of continuous; check whether $\lim_{x\to 0} f(x) = f(0)$. First $f(0) = 2(0)^2 + 0 = 0$. Second, to find the limit, since the function is split at $x = 0$, we check whether both 1-sided limits are equal.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x^2 + x = 0 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin 2x = \sin 0 = 0 \text{ so } \lim_{x\to 0} f(x) = 0.$$

Since $\lim_{x\to 0} f(x) = 0 = f(0)$, then f is continuous at $x = 0$.

b) To be differentiable at $x = 0$, both 1-sided derivatives must be equal.

$$\text{Right}: \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\sin(2x)-0}{x} = \lim_{x\to 0^+} \frac{2\sin(2x)}{2x} = 2\cdot 1 = 2.$$

$$\text{Left}: \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{2x^2+x-0}{x-0} = \lim_{x\to 0^+} 2x+1 = 1.$$

The two 1-sided derivatives are NOT equal, so f is NOT differentiable at $x = 0$.

11. Here are two methods. $\cot x = \frac{1}{\tan x}$. So

$$D_x(\cot x) = D_x\left(\frac{1}{\tan x}\right) = \frac{0\cdot \tan x - 1\cdot \sec^2}{\tan^2 x} = -\frac{\frac{1}{\cos^2}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

Or use $\cot x = \frac{\cos x}{\sin x}$. So

$$D_x(\cot x) = D_x\left(\frac{\cos}{\sin x}\right) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

12. a) $v(t) = s'(t) = e^t + te^t = (t+1)e^t$.
$a(t) = s''(t) = e^t + (t+1)e^t = (1+t+1)e^t = (t+2)e^t$.
$j(t) = s'''(t) = e^t + (t+2)e^t = (t+3)e^t$.

b) When $v(t) = e^t(t+1) = 0$. Since e^t is never 0, we need $t = -1$.

c) Notice the pattern above: The one-hundredth derivative would be $e^t(t+100) = e^t(t+100)$

13. Point $(1, f(1)) = (1, \sin\pi) = (1, 0)$. For the slope we need to calculate $f'(x) = \sin(\pi x) + x\cos(\pi x)\cdot\pi$. So $f'(1) = \sin\pi + \pi\cos\pi = -\pi$. The tangent is $y - 0 = -\pi(x-1)$ or $y = -\pi x + \pi$.

14. Also see previous labs for more practice.

a) $y' = \dfrac{x^{2/3}}{6} + \dfrac{2x^4}{5} + \dfrac{2x^{-2}}{3}$

b) $g'(x) = 3x^2\sec(x) + x^3\sec(x)\tan(x)$

c) $f'(x) = 2\sec(x^2)\tan(x^2)xe^{1+sec(x^2)}$

d) $g'(x) = \dfrac{4x}{(x^2+1)^2}$

e) $h'(t) = \dfrac{15t^2}{4(5t^3+1)^{3/4}}$

f) $g'(x) = -3\cos^2 x\sin x - 3\sin^2 x\cos x = -3\cos x\sin x(\cos x + \sin x)$

g) $g'(x) = 4(12 + xe^x)^3e^x(1+x)$

h) $f'(x) = e^x\tan(e^{3x}) + e^x\sec^2(e^{3x})3e^{3x} = e^x[\tan(e^{3x}) + 3e^{3x}\sec^2(e^{3x})]$

i

j) $p'(t) = e^{t^2+1} + te^{t^2+1}(2t) = (1+2t^2)e^{t^2+1}$

k) $f'(x) = -\dfrac{\sin x}{\cos x} = -\tan x$

l) $f'(t) = \dfrac{16t}{8t^2+1}$

m) $\dfrac{\frac{1}{t}\cdot t^3 - \ln t\cdot 3t^2}{(t^3)} = \dfrac{1-3\ln t}{t^4}$

n) $y = \ln|x^3 - x^2 + 4| - \ln|2x^3 - x|$; $y' = \dfrac{3x^2-2x}{x^3-x^2+4} + \dfrac{6x^2-1}{2x^3-x}$

o) $f(x) = \dfrac{1}{3}\ln(9x^2+11)$; $f'(x) = \dfrac{6x}{9x^2+11}$

p) $r'(t) = \dfrac{-2t+2t^2+1}{(2t^2-1)^2}$

15. **a)** Let $y = \ln x$. We want to find $\frac{dy}{dx}$. But $y = \ln x$. So $\underline{e^y = e^{\ln x}}$ or $\underline{e^y = x}$.

b) Take the derivative using the chain rule on the left: $D_x[e^y] = D_x[x]$ so $\underline{e^y \cdot \frac{dy}{dx} = 1}$.

c) Solve for $\frac{dy}{dx}$ to get: $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$.

16. **a)** $\dfrac{d}{dx}(x^3 - x^2y + y^2) = \dfrac{d}{dx}(13) \Rightarrow 3x^2 - 2xy - x^2\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$

$$\begin{aligned} &\Rightarrow \quad -x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2xy - 3x^2 \\ &\Rightarrow \quad (2y - x^2)\frac{dy}{dx} = 2xy - 3x^2 \\ &\Rightarrow \quad \frac{dy}{dx} = \frac{2xy - 3x^2}{2y - x^2} \end{aligned}$$

b) Point $(2, -1)$. Slope $m = \dfrac{dy}{dx} = \dfrac{2(2)(-1) - 3(2)^2}{2(-1) - (2)^2} = \dfrac{-16}{-6} = \dfrac{8}{3}$. Tangent: $y + 1 = \frac{8}{3}(x - 2)$.

c) $\dfrac{d}{dx}(x^2y^2 + e^{2y}) = \dfrac{d}{dx}(x) \Rightarrow 2xy^2 + 2x^2y\dfrac{dy}{dx} + 2e^{2y}\dfrac{dy}{dx} = 1$

$$\begin{aligned} &\Rightarrow \quad 2x^2y\frac{dy}{dx} + 2e^{2y}\frac{dy}{dx} = 1 - 2xy^2 \\ &\Rightarrow \quad (2x^2y + 2e^{2y})\frac{dy}{dx} = 1 - 2xy^2 \\ &\Rightarrow \quad \frac{dy}{dx} = \frac{1 - 2xy^2}{2e^{2y} + 2x^2y} \end{aligned}$$

17. This problem tests whether you understand the chain rule. Quickly compute:

a) $\dfrac{d}{dx}[4\sec(x)] = 4\sec(x)\tan(x)$

b) $\dfrac{d}{dx}[\sec(4x)] = 4\sec(4x)\tan(4x)$

c) $\dfrac{d}{dx}\left[\sec(\frac{x}{4})\right] = \frac{1}{4}\sec(\frac{x}{4})\tan(\frac{x}{4})$

d) $\dfrac{d}{dx}\left[\dfrac{\sec x}{4}\right] = \frac{1}{4}\sec x \tan x$

e) $\dfrac{d}{dx}[\sec(x^4)] = 4x^3\sec(x^4)\tan(x^4)$

f) $\dfrac{d}{dx}[\sec^4 x] = 4\sec^3 x \sec x \tan x = 4\sec^4 x \tan x$

18. Implicit differentiation (brief answers):

a) $3\dfrac{dy}{dx} + \dfrac{1}{y}\dfrac{dy}{dx} = 2x + 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{2x + 1}{3 + \frac{1}{y}}$

b) $2y\dfrac{dy}{dx} - 2xy - x^2\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{2xy}{2y - x^2}$

c) $e^{xy^2}\left(y^2 + 2xy\dfrac{dy}{dx}\right) = 2x \Rightarrow \dfrac{dy}{dx} = \dfrac{2x - y^2e^{xy^2}}{2xye^{xy^2}}$

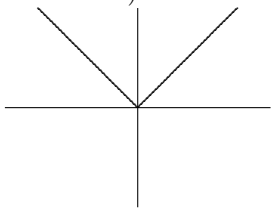
d) $2x + 2xy^2 + 2x^2y\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{2x + 2xy^2}{2x^2y + 3y^2}$

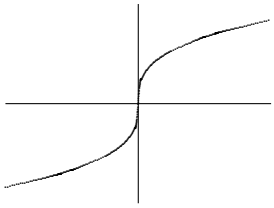
e) $y^2 + 2xy\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y^2}{2xy}$

19. True or false. If false, draw a function which illustrates your answer.

a) If f is differentiable at $x = a$, then f is continuous at $x = a$. True; this is a theorem we proved.

b) If f is continuous at $x = a$, then f is differentiable at $x = a$. False... a function like $y = |x|$ is continuous but not differentiable at $x = 0$ because it is unbroken (continuous) but has a corner (not differentiable). Or a function may have a vertical tangent like $y = \sqrt[3]{x}$ does at 0. The curve is unbroken but there is no well-defined slope there. (See your notes! or the text.)



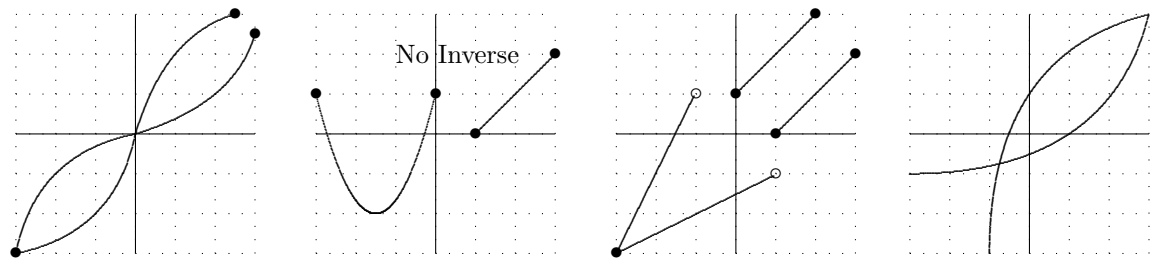


c) If f is not continuous at $x = a$, then f is not differentiable at a. True—another way of saying part (a).

20. **a)** g is the **inverse of** f if $g(f(x)) = x$ for all x in the domain of f AND $f(g(x)) = x$ for all x in the domain of g.

b) f is a **one-to-one** function if it never has the same output value twice, that is, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

21. Explain which of these functions have inverses and which do not. If a function has an inverse, graph it on the same set of axes.

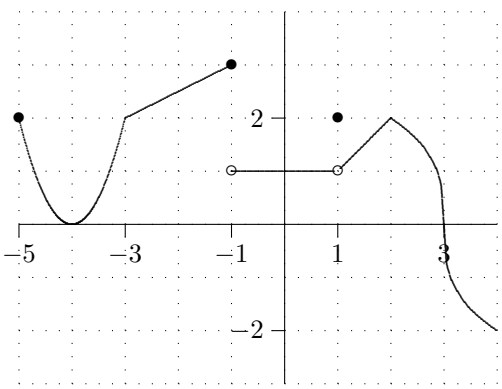


22. **a)** See the table.

b) The points where f is *not continuous*: $x = -1$ and 1.

c) $x = 3, 2, -3$ are the points where f is *not differentiable* even though f is continuous.

d) Impossible. Differentiable implies continuous.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$	$f(a)$	Ctns	Diff'ble
-4	0	0	0	0	Yes	Yes
-3	2	2	2	2	Yes	No
-2	2.5	2.5	2.5	2.5	Yes	Yes
-1	3	1	DNE	3	No	No
0	1	1	1	1	Yes	Yes
1	1	1	1	2	No	No
2	2	2	2	2	No	No
3	0	0	0	0	Yes	No
5	3	3	3	3	Yes	Yes
8	4	4	4	Not 4	No	No

23. The slope is 0 at this low point. So we need to set $f'(x) = 0$ and solve for x.

$$f'(x) = 1 \ln x + x \cdot \tfrac{1}{x} = \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}.$$

So $x = e^{-1}$ and $y = f(e^{-1}) = e^{-1} \ln e^{-1} = e^{-1}(-1) = -e^{-1}$. So the point is $(-1, -e^{-1})$.

24. **a)** You can evaluate this as in Problem 7(g), using the derivative at a point.

$$\lim_{h\to 0} \frac{\overbrace{e^h}^{f(0+h)} - \overbrace{e^0}^{f(0)}}{h} \overset{\text{definition}}{=} f'(0) = D_x(e^x)\Big|_{x=0} = e^x\Big|_{x=0} = e^0 = 1.$$

b) $\displaystyle\lim_{h\to 0} \frac{e^{2h} - 1}{h} = \lim_{h\to 0} \frac{(e^h + 1)(e^h - 1)}{h} = \lim_{h\to 0} (e^h + 1) \cdot \frac{(e^h - 1)}{h} = 2 \cdot 1 = 2.$

25. The u's are for the inverse trig functions.

a) $\displaystyle\frac{d}{dx}(\arctan(6x^2)) = \frac{1}{1 + 36x^4} \cdot 12x = \frac{12x}{1 + 36x^4}$ $\qquad (u = 6x^2)$

b) $\displaystyle\frac{d}{dx}(\arcsin(\sqrt{x})) = \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}\sqrt{1 - x}}$ $\qquad (u = \sqrt{x})$

c) $\displaystyle\frac{d}{dx}(\arctan(e^{2x})) = \frac{2e^{2x}}{1 + e^{4x}}$ $\qquad (u = e^{2x})$

d) $\displaystyle\frac{d}{dx}(\arcsin(\arcsin x)) = \frac{1}{\sqrt{1 - (\arcsin x)^2}} \cdot \frac{1}{\sqrt{1 - x^2}}$ $\qquad (u = \arcsin x))$

e) $\displaystyle\frac{d}{dx}[\arctan(\ln|6x|)] = \frac{1}{1 + (\ln|6x|)^2} \cdot \frac{1}{6x} \cdot 6 = \frac{1}{x[1 + (\ln|6x|)^2]}$ $\qquad (u = \ln|6x|)$

f) $\dfrac{d}{dx}(\arcsin(6e^{\sin x})) = \dfrac{1}{\sqrt{1-(6e^{\sin x})^2}} \cdot (6e^{\sin x})(\cos x) = \dfrac{6\cos x e^{\sin x}}{\sqrt{1-(6e^{\sin x})^2}}$ $\qquad (u = 6e^{\sin x})$

g) $\dfrac{d}{dx}(e^{2\arcsin x^2}) = (e^{2\arcsin x^2}) \cdot 2 \cdot \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \dfrac{4xe^{2\arcsin x^2}}{\sqrt{1-x^4}}$ $\qquad (u = x^2)$

h) $\dfrac{d}{dx}[\arcsin 2x(\tan 5x^2)] = \dfrac{2\tan 5x^2}{\sqrt{1-4x^2}} + (\arcsin 2x)10x\sec^2(5x^2)$ $\qquad (u = 2x,\ 5x^2)$

i) $\dfrac{d}{dx}(\ln|\arctan e^{x^4+1}|) = \dfrac{1}{\arctan e^{x^4+1}} \cdot \dfrac{1}{1+(e^{x^4+1})^2} \cdot e^{x^4+1} \cdot 4x^3 = \dfrac{4x^3e^{x^4+1}}{(\arctan e^{x^4+1})(1+e^{2x^4+2})}$