

Calc 1 Review

Simplifying Rational Expressions

- ⇒ factor top and bottom, then cancel $\frac{(x^2 + 5x + 6)}{(x+3)} = \frac{(x+2)(x+3)}{(x+3)} = (x+2)$
- ⇒ always try and GCF factor first, then factor the remaining terms
- ⇒ use conjugate to simplify expressions with radicals

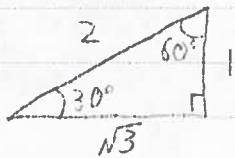
$$\frac{(x-4)}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)} = \frac{(x-4)(\sqrt{x}-2)}{(x-4)} = \sqrt{x}-2$$
- ⇒ when multiplying by the conjugate you should have no "cross terms" ⇒ $(\sqrt{x}+2)(\sqrt{x}-2) = x+2\sqrt{x}-2\sqrt{x}-4 = x-4$
- ⇒ if everything is not factored, you can not cancel yet
 $\frac{x^2+5}{x} \neq \frac{x+5}{1}$, $\frac{x+5}{x}$ cannot be simplified like this
- ⇒ difference of perfect squares can factor, $(y^2 - 36) = (y+6)(y-6)$
- ⇒ sum of perfect squares cannot factor, $x^2 + 4 \neq (x+2)(x-2)$
- ⇒ distribute negatives! $x - (1+x) = x - 1 - x = -1$

Irig Functions

- ⇒ convert radians to degrees, degrees to radians (π radians = 180°)
- ⇒ know this chart

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin \theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos \theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan \theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	DNE

- ⇒ and this triangle



⇒ know these definitions $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\sec \theta = \frac{1}{\cos \theta}$, $\csc \theta = \frac{1}{\sin \theta}$

⇒ know $\sin(x)$ & $\cos(x)$ are continuous, and oscillate between -1 and 1

⇒ know $\sin^2 \theta + \cos^2 \theta = 1$ for any value of θ

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Solving absolute Value

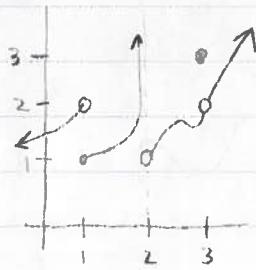
- ⇒ know how to convert $|x-4| < 7$ to $-7 < (x-4) < 7 \Rightarrow -3 < x < 11$
- ⇒ know $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ and $\sqrt{x^2} = |x|$
- ⇒ know that $\lim_{x \rightarrow \infty} \frac{x}{|x|} = 1$ but $\lim_{x \rightarrow -\infty} \frac{x}{|x|} = -1$
- ⇒ know $|2x+3| > 9$ means $2x+3 > 9$ or $2x+3 < -9$
and each must be solved separately $\Rightarrow x > 6$ or $x < -6$

Average / Instantaneous Velocity

- ⇒ know that average velocity = average rate of change of position
 $v_{\text{avg}} = \frac{s(t) - s(2)}{t - 2}$ = average velocity on the interval $[2, t]$
could be something besides
- $v_{\text{inst.}} = \lim_{t \rightarrow 2} \left(\frac{s(t) - s(2)}{t - 2} \right)$ = instantaneous velocity at the point $t=2$
- $v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(t) - s(5)}{t - 5}$ = average velocity on $[5, t]$

⇒ average velocity is defined on an interval, instantaneous velocity is defined at a point

Limits (Graphically)



Intervals of Continuity

$$(-\infty, 1) \cup (1, 2) \cup (2, 3) \cup (3, \infty)$$

must exclude $x=1, 2, 3$

at $x=1$, $\lim_{x \rightarrow 1^-} f(x) = 2$, $\lim_{x \rightarrow 1^+} f(x) = 1$
 so $\lim_{x \rightarrow 1} f(x)$ DNE, No RD, No VA

at $x=2$, $\lim_{x \rightarrow 2^-} f(x) = \infty$, $\lim_{x \rightarrow 2^+} f(x) = 1$
 so $\lim_{x \rightarrow 2} f(x)$ DNE, No RD, but Yes VA

at $x=3$, $\lim_{x \rightarrow 3^-} f(x) = 2$, $\lim_{x \rightarrow 3^+} f(x) = 2$
 so $\lim_{x \rightarrow 3} f(x) = 2$, so Yes RD, No VA

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Limits with Polynomials

Polynomials are continuous \Rightarrow PLUG IN!! if f is cont., then $\lim_{x \rightarrow 2} f(x) = f(2)$ by the definition of continuous functions

$$\lim_{x \rightarrow 5} (x^2 + 3x - 20) = 5^2 + 3(5) - 20 = 25 + 15 - 20 = 20$$

if my limit is going to $\pm\infty$ use Highest power Rule

$$\lim_{x \rightarrow \infty} (x^5 - 7x^9 - 10) = \lim_{x \rightarrow \infty} (-7x^9) = -\infty \Rightarrow \text{watch negative sign}$$

Limits with Rationals (limit not going to $\pm\infty$)

With Rationals and limits, try and plug in. there are 3 cases

1. you get $(\#/\#)$, or a number over $\overset{\neq 0}{\text{a number}}$. this is your answer, no more work required

$$\text{Ex: } \lim_{x \rightarrow 2} \frac{x^2 + 5x + 7}{x - 3} = \frac{2^2 + 5(2) + 7}{2 - 3} = \frac{4 + 10 + 7}{-1} = \frac{21}{-1} = -21$$

2. you get $(0/0) \Rightarrow$ this is indeterminate form. you must factor and do more work to find this limit

$$\text{Ex: } \lim_{x \rightarrow 5} \frac{25 - x^2}{(x - 5)} = \frac{0}{0} \Rightarrow \text{more work} \Rightarrow \lim_{x \rightarrow 5} \frac{(5+x)(5-x)}{(x-5)} = \underbrace{\lim_{x \rightarrow 5} -(5+x)}_{\text{now plug in}} = -10$$

3. you get $(\#/\#)$ \Rightarrow you are going to ∞ or $-\infty$. you have to do one-sided limits, and determine if you go to $+\infty$ or $-\infty$ by keeping track of signs

$$\text{Ex: } \lim_{x \rightarrow 2} \frac{x^2 + 5}{x^2 - 2x} = \frac{9}{0} \Rightarrow \text{going to } + \text{ or } -\infty \Rightarrow \text{do 1 sided limits}$$

$$\lim_{x \rightarrow 2^+} \frac{x^2 + 5}{x(x-2)} = -\infty, \quad \lim_{x \rightarrow 2^-} \frac{x^2 + 5}{x(x-2)} = \infty \Rightarrow \infty \neq -\infty, \text{ so } \lim_{x \rightarrow 2} \frac{x^2 + 5}{x^2 - 2x} \text{ DNE}$$

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Limit Properties

- ⇒ know sum, difference, power, quotient, composition properties
- ⇒ know how to formally write these problems
- ⇒ basically, know when you can plug in and when you can't

$$\text{Ex: } \lim_{x \rightarrow 2} \frac{x^2 + 5 \cos(\pi x)}{e^x} = \frac{\lim_{x \rightarrow 2} (x^2 + 5 \cos(\pi x))}{\lim_{x \rightarrow 2} (e^x)} = \frac{\lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 5 \cos(\pi x)}{\lim_{x \rightarrow 2} e^x}$$

$$= \frac{(\lim_{x \rightarrow 2} x)^2 + 5 \cos(\lim_{x \rightarrow 2} \pi x)}{e^{\lim_{x \rightarrow 2} (x)}} = \frac{2^2 + 5 \cos(2\pi)}{e^2} = \frac{4 + 5(1)}{e^2} = \boxed{\frac{9}{e^2}}$$

* basically, we justify why we can plug in *

Highest Power Rule (w/ rational functions)

- * can only be used when $x \rightarrow \pm\infty$, not when $x \rightarrow a$ where a is some finite number
- ⇒ take the highest power of x in the top and bottom and proceed from there

$$\text{Ex: } \lim_{x \rightarrow -\infty} \frac{5x^3 + 5x^2 + 10x^5}{3x - 7x^5} \stackrel{\text{HP}}{=} \lim_{x \rightarrow -\infty} \frac{10x^5}{-7x^5} = \lim_{x \rightarrow -\infty} \frac{10}{-7} = \frac{10}{-7}$$

- ⇒ if the power up top is higher, we go to $+\infty$ or $-\infty$
- ⇒ if the powers are the same, we go to some #
- ⇒ if the power in the bottom is higher, we go to 0

Horizontal Asymptotes

- ⇒ to check for a Horizontal Asymptote, take $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$
- ⇒ if $\lim_{x \rightarrow \infty} f(x) = L$ (L is any Finite #), then there is a Horizontal Asymptote at $y = L$ ($L = 0$ is allowed)
- ⇒ if $\lim_{x \rightarrow \infty} f(x) = \infty$, there is No horizontal asymptote

$$\text{Ex: } \lim_{x \rightarrow \infty} \frac{3x^2 + x}{4x^3} = \stackrel{\text{HP}}{\lim_{x \rightarrow \infty}} \frac{3x^2}{4x^3} = \lim_{x \rightarrow \infty} \frac{3}{4x} = 0 \Rightarrow \text{HA at } y = 0$$

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Vertical Asymptotes (rational functions)

- ⇒ to check for a vertical asymptote, check all the points where $f(x)$ is discontinuous, then take the limit towards each of these points
- ⇒ if $f(x)$ is a rational function, this means that we need to check all the points where the bottom is 0
- ⇒ need to take 1-sided limits towards these points, ∞ means VA

Ex: $f(x) = \frac{4-x^2}{x^2+2x} \Rightarrow$ factoring, $f(x) = \frac{(2+x)(2-x)}{x(x+2)}$ ⇒ $f(x)$ is discontinuous at $x=0, x=-2$. check these points

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{(2+x)(2-x)}{x(x+2)} = \lim_{x \rightarrow 0^+} \frac{\cancel{(2+x)}(2-x)}{\cancel{x}(x+2)} = \frac{2}{0^+} = \infty \Rightarrow \text{VA at } x=0$$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{(2+x)(2-x)}{x(x+2)} = \lim_{x \rightarrow -2^-} \frac{(2-x)}{\cancel{x}} = \frac{4}{-2} = -2 \Rightarrow \text{No VA at } x=-2$$

- * For a VA, you need an unmatched factor in the bottom. $x/(x+2)$ will have a VA at $x=-2$
- $x(x+2)/(x+2)$ will not have a VA at $x=-2$

Removable Discontinuities (rational functions)

- ⇒ to check for a Removable Discontinuity, check all the points where $f(x)$ is discontinuous, then take the limit towards each of these points

⇒ if $f(x)$ is a rational function, find all the points where the bottom is 0

- ⇒ for a Removable Discontinuity, we need two things to happen
 1. $\lim_{x \rightarrow a} f(x)$ exists, and is not $\pm\infty$ (this is a two sided limit)
 2. $\lim_{x \rightarrow a} f(x) \neq f(a)$. $f(a)$ may not even exist

Ex: $f(x) = \frac{x^2(x+2)}{x} \Rightarrow$ check $x=0 \Rightarrow \lim_{x \rightarrow 0} \frac{x^2(x+2)}{x} = \lim_{x \rightarrow 0} x(x+2) = 0(0+2) = 0$

$f(0)$ DNE, so since $\lim_{x \rightarrow 0} f(x) \neq f(0)$, we have a RD at $x=0$

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Removable Discontinuities V.S. Vertical Asymptotes

→ a Removable Discontinuity has a matched factor in the top and bottom, causing cancellation and the limit to exist

→ a Vertical Asymptote has an unmatched factor in the top and bottom, causing only the bottom to go to 0 and the limit to be $\pm\infty$

Ex: $f(x) = \frac{(x+2)}{(x+2)(x-7)}$ → $(x+2)$ is matched, so $x = -2$ is a RD
 $\Rightarrow (x-7)$ is unmatched, so $x = 7$ is a VA
 * in both cases, $f(-2)$ and $f(7)$ DNE

Continuity Checklist

→ for $f(x)$ to be continuous at the point $x = a$, we need to satisfy 3 conditions

- 1. $f(a)$ exists
 - 2. $\lim_{x \rightarrow a} f(x)$ exists, and is not $\pm\infty$
 - 3. $\lim_{x \rightarrow a} f(x) = f(a)$
- } memorize this.

Ex: $f(x) = \begin{cases} 2x & \text{if } x > 0 \\ x^2 & \text{if } x \leq 0 \end{cases}$ is $f(x)$ continuous at $x = 0$?

1. $f(0) = 0^2 = 0$, so $f(0)$ exists ✓

2. f is defined piecewise, so we must use 1-sided limit
 $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0$, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x = 0$, so because

$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$, the two sided limit $\lim_{x \rightarrow 0} f(x)$ exists ✓

3. $\lim_{x \rightarrow 0} f(x) = 0 \neq f(0) = 0$, so $\lim_{x \rightarrow 0} f(x) \neq f(0)$ ✓

so $f(x)$ is continuous at $x = 0$

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Definitions

Know these formal definitions

Horizontal Asymptote: The line $y=L$ is a horizontal Asymptote of the function f if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$

Vertical Asymptote: The line $x=a$ is a vertical Asymptote of the function f if $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^-} f(x) = \pm\infty$

Left Continuous: $f(x)$ is left continuous at $x=a$ if $f(a)$ exists, $\lim_{x \rightarrow a^-} f(x)$ exists, and $\lim_{x \rightarrow a^-} f(x) = f(a)$

Right Continuous: $f(x)$ is right continuous at $x=a$ if $f(a)$ exists, $\lim_{x \rightarrow a^+} f(x)$ exists, and $\lim_{x \rightarrow a^+} f(x) = f(a)$

Continuous: $f(x)$ is continuous at $x=a$ if $f(a)$ exists, $\lim_{x \rightarrow a} f(x)$ exists, and $\lim_{x \rightarrow a} f(x) = f(a)$

Removable Discontinuity: $f(x)$ has a removable discontinuity at $x=a$ if $\lim_{x \rightarrow a} f(x)$ exists ($\neq \infty$) and $\lim_{x \rightarrow a} f(x) \neq f(a)$

ϵ - δ Limits * One of these WILL be on your exam*

⇒ know the formal definition of the limit

Def: $\lim_{x \rightarrow a} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ so that when $0 < |x-a| < \delta$, we have $|f(x) - L| < \epsilon$

Ex: show $\lim_{x \rightarrow 3} 2x + 5 = 11$ with ϵ, δ

$$\Rightarrow |(2x+5) - 11| < \epsilon \Rightarrow |2x - 6| < \epsilon \Rightarrow |2(x-3)| < \epsilon \Rightarrow 2|x-3| < \epsilon$$

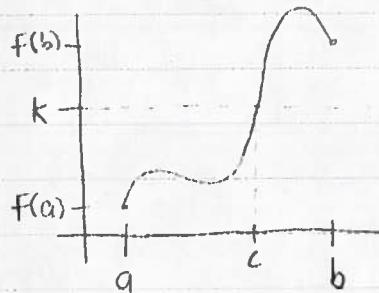
$$\Rightarrow |x-3| < \frac{\epsilon}{2}, \text{ so pick } \boxed{\delta = \frac{\epsilon}{2}}$$

*this is your scapework. Formally, now show that $0 < |x-3| < \frac{\epsilon}{2}$ implies $|2x+5-11| < \epsilon$

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Intermediate Value Theorem

- ⇒ know what this theorem means with pictures
- ⇒ Formally, if f is continuous on $[a, b]$ and k is some value such that $f(a) < k < f(b)$, then there is at least 1 value, c , where $a < c < b$ such that $f(c) = k$



* We are never trying to find a specific c value, we just need to know one exists

Ex: Show that $x^2 - 2 = 0$ has a solution on the interval $[0, 2]$ using the IVT

⇒ $f(x) = x^2 - 2$ is continuous because it is a polynomial
so $f(x)$ is continuous on $[0, 2]$

$$\Rightarrow f(0) = 0^2 - 2 = -2, \quad f(2) = 2^2 - 2 = 2, \quad \text{so since } -2 < 0 < 2$$

there is some value c on $[0, 2]$ such that

$$f(c) = 0$$