## 5-Minute Review: Rational Functions

**DEFINITION 4.-1.1.** A **rational function** is a function of the form

$$y = r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials. The domain of a rational function consists of all values of $x$ such that $q(x) \neq 0$.

Here the term 'rational' means 'ratio' as in the ratio of two polynomials.

**EXAMPLE 4.-1.2.** Here are three examples. Notice the last one requires rewriting.

$$(a)\ r(x) = \frac{2x^2+3x+1}{4x^{11}+9x^2} \qquad (b)\ s(x) = \frac{1}{2x+7} \qquad (c)\ r(t) = 6(t+1)t^{-2} = \frac{6t+6}{t^2}$$

And some non-examples (because of the sine function, root, and fractional exponent):

$$(a)\ r(x) = \frac{\sin(2x^2+1)}{4x^{11}+9x^2} \qquad (b)\ s(x) = \sqrt{\frac{3x}{2x+7}} \qquad (c)\ t(x) = \frac{3x^{1/2}+1}{x^2+4}$$

*Caution:* We must be careful when simplifying expressions involving rational functions. Do you see why the functions below are NOT the same function?

$$r(x) = \frac{x^2-4}{x^2-2x} = \frac{(x-2)(x+2)}{x(x-2)} \text{ and } s(x) = \frac{x+2}{x}$$

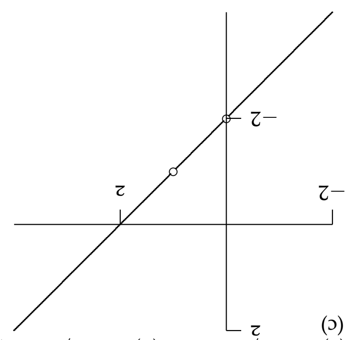
The functions are different because their domains are different. The domain of $r(x)$ is all $x \neq 0, 2$ while the domain of $s(x)$ is all $x \neq 0$.

**YOU TRY IT 4.1.** (*a*) Determine all $x$ for which $\frac{x^2-4x-5}{x^2-1}$ and $\frac{x-5}{x-1}$ have the same values.

(*b*) Do the same for $\frac{x^3-3x^2+2x}{x^2-x}$ and $x-2$

(*c*) Carefully sketch the graph of $\frac{x^3-3x^2+2x}{x^2-x}$. Think about part (b).

ANSWER TO YOU TRY IT 4.1. (a) $x \neq \pm 1$. (b) $x \neq 0, 1$. (c)




**YOU TRY IT 4.2.** Which are of the following functions are polynomials (state the degree)? Rational (but not a polynomial)? Neither?

$(a)\ p(x) = -4x^3+2x+11$ $\quad (b)\ r(x) = \frac{2x^2+3x+1}{4x^{11}+9x^2}$ $\quad (c)\ q(x) = \frac{1}{5x^2} - \frac{7}{x}$

$(d)\ s(x) = \frac{1}{2x+7}$ $\quad (e)\ r(t) = \frac{t^2+1}{t^2-1}$ $\quad (f)\ p(x) = \sin(x^2+1)$

$(g)\ s(x) = 2x^2 - x^{1/2} + 7$ $\quad (h)\ q(t) = \sqrt{t^3+t^2+1}$ $\quad (i)\ r(x) = 11$

$(j)\ r(x) = 3^{1/2}x^4 - 2x + 6$ $\quad (k)\ s(x) = \sqrt{\frac{3x}{2x+7}}$ $\quad (l)\ t(x) = \frac{3x^{1/2}+1}{x^2+4}$

$(m)\ f(x) = -\frac{2}{3}x^5 + 3x^4 + x^2 - 11$ $\quad (n)\ p(x) = 5x^2 - x^{1/3} - 23$ $\quad (o)\ g(x) = 6x^{-2} + 4x^{-1} + 2$

$(p)\ q(x) = 3x - 4x^2 + \frac{x^3}{6}$

ANSWER TO YOU TRY IT 4.2. Polynomial: a (3), i (0), j (4), m (5), and p (3). Rational (but not a polynomial: b, c d, e, and o. Neither: f, g, h, k, l, and n.

## 4.0 5-Minute Review: Conjugates

Conjugates are usually discussed in reference to expressions involving square roots and typically have the form $a + \sqrt{b}$ and $a - \sqrt{b}$, where $a$ can be any expression. For example, $\sqrt{x} + \sqrt{3}$ and $\sqrt{x} - \sqrt{3}$. Conjugates are useful because when you multiply them together, the 'middle terms' cancel: e.g.,

$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

or

$$(\sqrt{x} + \sqrt{3})(\sqrt{x} - \sqrt{3}) = x - 3.$$

**EXAMPLE 4.0.1.** Use conjugates to simplify the following expression: $\dfrac{4}{\sqrt{x+2} - \sqrt{x}}$.

**Solution.** Multiply both the numerator and denominator by the conjugate:

$$\frac{4}{\sqrt{x+2} - \sqrt{x}} \cdot \frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}} = \frac{4(\sqrt{x+2} + \sqrt{x})}{x+2-x} = \frac{4(\sqrt{x+2} + \sqrt{x})}{2} = 2(\sqrt{x+2} + \sqrt{x})$$

**EXAMPLE 4.0.2.** Use conjugates to simplify the following expression: $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

**Solution.** Multiply both the numerator and denominator by the conjugate:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

**YOU TRY IT 4.3.** Simplify each of these expressions by using an appropriate conjugate.

(a) $\dfrac{x-5}{\sqrt{x} - \sqrt{5}}$ (b) $\dfrac{2x-18}{\sqrt{x} - 3}$ (c) $\dfrac{\sqrt{8+h} - \sqrt{8}}{h}$

ANSWER TO YOU TRY IT 4.3. (a) $\sqrt{x} + 5$; (b) $2(\sqrt{x} + 3)$; (c) $\frac{1}{\sqrt{8+h}+\sqrt{8}}$

# *Calculating Limits*

## 4.1 *Basic Limit Properties*

There are several basic limit properties which ease the problem of calculating limits. Most of these properties are quite straightforward and would be what you might suspect is true. In more advanced calculus courses (called analysis courses) each of these properties would be carefully proven.

The first property that we list is almost a tautology.

**THEOREM 4.1.1.** $\lim_{x \to a} x = a.$

The limit is asking: 'As $x$ approaches $a$, what is $y = f(x) = x$ approaching?' Well, obviously $a$! This is also geometrically clear in the left-hand panel of Figure 4.1.

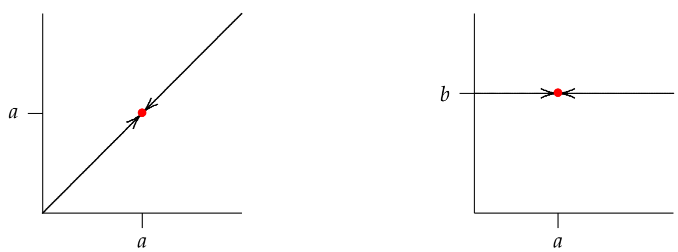



Figure 4.1: Left: 'As $x$ approaches $a$, $x$ approaches $a$' so $\lim_{x \to a} x = a$. Right: For 'As $x$ approaches $a$, the constant function $f(x) = b$ is always equal to $b$' so $\lim_{x \to a} b = b$.

**THEOREM 4.1.2.** If $b$ is any constant, then $\lim_{x \to a} b = b$.

The limit is asking: 'As $x$ approaches $a$, what is $y = f(x) = b$ approaching?' Well, since $b$ is constant, $f(x)$ is approaching $b$, it is actually equal to $b$. This is geometrically clear in the right-hand panel of Figure 4.1. We can combine the first two limit theorems in a more general result about lines or linear functions.

**THEOREM 4.1.3 (Linear functions).** Let $a$, $b$, and $m$ be any constants and let $f$ be the linear function $f(x) = mx + b$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} mx + b = ma + b = f(a).$$

Remember from Lab 1, that a function where $\lim_{x \to a} f(x) = f(a)$ is called a **continuous** function at $a$.

Think about what this says: as $x \to a$, $mx \to ma$, so $mx + b \to ma + b$. We have actually used two mathematical operations, multiplication and addition, and said that the limit operation interacts with them in a special way: whether the limit is computed before or after the operation does not matter. We explore this further in the next section.

**EXAMPLE 4.1.4.** Determine the following limits.

(*a*) $\lim_{x\to 4} f(x)$ where $f(x) = -2x + \frac{1}{2}$.

(*b*) $\lim_{t\to -2} g(t)$ where $g(t) = \frac{7}{2}t - 8$.

(*c*) $\lim_{x\to -1} h(x)$ where $h(x) = 9$.

**Solution.**

(*a*) $\lim_{x\to 4} f(x) = \lim_{x\to 4} \left(-2x + \frac{1}{2}\right) \overset{\text{Linear}}{=} f(4) = \dfrac{15}{2}$.

(*b*) $\lim_{t\to -2} g(t) = \lim_{t\to -2} \left(\frac{7}{2}t - 8\right) \overset{\text{Linear}}{=} g(-2) = -15$.

(*c*) $\lim_{x\to -1} h(x) = \lim_{x\to -1} 9 \overset{\text{Constant}}{=} 9$.

## *Properties Involving the Order of Operations*

The next several properties involve the interchange of the limit operation with other arithmetic operations such as addition or multiplication. These properties are important because they greatly simplify the calculation of limits. It is important to recognize that not all mathematical operations can be interchanged in this way. For example, the square root of a sum is not the sum of the square roots:

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}.$$

In particular, $\sqrt{9+16} = 5$ while $\sqrt{9} + \sqrt{16} = 7$. Clearly, the order of operations matters; taking the square root before or after summing changes the result.

However, the limit properties below indicate that certain mathematical operations with limits have the same answer, regardless of the order in which they are carried out. This is special and useful! Note the order of operations in each part.

**THEOREM 4.1.5** (Order of Operations). Assume that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and that $b$ is a constant. Then

1. **(Constant Multiple)**. $\lim_{x\to a} bf(x) = b(\lim_{x\to a} f(x)) = bL$.
2. **(Sum or Difference)**. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$. 'The limit of a sum is the sum of the limits.'
3. **(Product)**. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x)) \cdot (\lim_{x\to a} g(x)) = LM$. 'The limit of a product is the product of the limits.'
4. **(Quotient)**. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, as long as $M \neq 0$. 'The limit of a quotient is the quotient of the limits.'

**EXAMPLE 4.1.6.** Determine the following limits. Indicate the limit properties used at each step. Assume that $\lim_{x\to 2} f(x) = 4$ and $\lim_{x\to 2} g(x) = -3$. Evaluate

(*a*) $\lim_{x\to 2}[2f(x) + 3g(x)] \overset{\text{Sum}}{=} \lim_{x\to 2} 2f(x) + \lim_{x\to 2} 3g(x) \overset{\text{Const Mult}}{=} 2\lim_{x\to 2} f(x) + 3\lim_{x\to 2} g(x) = 2(4) + 3(-3) = -1$.

(*b*) $\lim_{x\to 2} -2f(x)g(x) \overset{\text{Prod}}{=} \lim_{x\to 2} -2f(x) \cdot \lim_{x\to 2} g(x) \overset{\text{Const Mult}}{=} -2\lim_{x\to 2} f(x) \cdot \lim_{x\to 2} g(x) = -2(4)\cdot(-3) = 24$.

(*c*) $\lim_{x\to 2} \dfrac{6x+7-f(x)}{g(x)} \overset{\text{Diff, Quot}}{=} \dfrac{\lim_{x\to 2}(6x+7) - \lim_{x\to 2} f(x)}{\lim_{x\to 2} g(x)} \overset{\text{Linear}}{=} \dfrac{19-4}{-3} = -5$.

(*d*) $\lim_{x\to a} x^2 = \lim_{x\to a} x \cdot x \overset{\text{Prod}}{=} \lim_{x\to a} x \cdot \lim_{x\to a} x \overset{\text{Thm 1}}{=} a \cdot a = a^2$. More generally, if $n$ is a positive integer, then $\lim_{x\to a} x^n = a^n$ by using the product limit law. That means the the function $f(x) = x^n$ is *continuous* for every point $a$.

*More Properties of Limits: Order of Operations*

**THEOREM 4.1.7** (Order of Operations, Continued)**.** Assume that $\lim_{x\to a} f(x) = L$ and that $m$ and $n$ are positive integers. Then

5. **(Power)**. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = L^n$.
6. **(Fractional Power)**. Assume that $\frac{n}{m}$ is reduced. Then

$$\lim_{x\to a}[f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m} = L^{n/m},$$

provided that $f(x) \geq 0$ for $x$ near $a$ if $m$ is even.

**EXAMPLE 4.1.8.** Determine $\lim_{x\to 3}(4x-10)^5$. Indicate the limit properties used at each step.

**Solution.** $\lim_{x\to 3}(4x-10)^5 \overset{\text{Powers}}{=} [\lim_{x\to 3} 4x - 10]^5 \overset{\text{Linear}}{=} [4(3)-10]^5 = (2)^5 = 32.$

**EXAMPLE 4.1.9.** Determine $\lim_{x\to 2}\sqrt{5x-1}$. Indicate the limit properties used at each step.

**Solution.** Notice that $\sqrt{5x-1} = (5x-1)^{1/2}$ is a fractional power function. In the language of Theorem 4.1.7, $\frac{n}{m} = \frac{1}{2}$ is reduced and $m = 2$ is even. Near $x = 2$, $f(x) = 5x-1$ is positive. Theorem 4.1.7 applies and we may calculate the limit as

$$\lim_{x\to 2}\sqrt{5x-1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to 2} 5x-1\right)^{1/2} \overset{\text{Linear}}{=} (9)^{1/2} = 3.$$

**EXAMPLE 4.1.10.** Determine $\lim_{x\to -3}(10x+3)^{4/3}$. Indicate the limit properties used at each step.

**Solution.** $(10x+3)^{4/3}$ is a fractional power function with $\frac{n}{m} = \frac{4}{3}$ which is reduced and $m = 3$ is odd. So Theorem 4.1.7 applies and we may calculate the limit as

$$\lim_{x\to -3}(10x+3)^{4/3} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to -3} 10x+3\right)^{4/3} \overset{\text{Linear}}{=} (-30+3)^{4/3} = (-27)^{4/3} = 81.$$

**EXAMPLE 4.1.11.** Determine $\lim_{x\to -3}(x^2-25)^{3/4}$.

**Solution.** $(x^2-25)^{3/4}$ is a fractional power function. In the language of Theorem 4.1.7, $\frac{n}{m} = \frac{3}{4}$ is reduced and $m = 4$ is even. Near $x = -3$, $x^2 - 25$ is *negative*. Since $(x^2-25)^{3/4}$ is not even defined near $-3$, this limit does not exist.

**THEOREM 4.1.12** (Special Functions)**.** Let $n$ be a positive integer and $c$ be any constant.

7. **(Monomials)**. $\lim_{x\to a} cx^n = ca^n$.
8. **(Polynomials)**. If $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a degree $n$ polynomial, then

$$\lim_{x\to a} p(x) = p(a).$$

9. **(Rational Functions)**. If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function, then for any point $a$ in the domain of $r(x)$

$$\lim_{x\to a} r(x) = r(a).$$

Theorem 4.1.12 says that the limit of polynomial or rational function as $x \to a$ is the same as the value of the function at $x = a$. This is not true of all limits. For example, we saw that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, yet we can't even put $x = 0$ into this function! Those special or 'nice' functions where $\lim_{x\to a} f(x) = f(a)$ are called **continuous** at $x = a$. We will examine them in depth in a few days. For the moment we can say that polynomials are continuous everywhere and rational functions are continuous at every point in their domains.

*Proof.* Let's see how limit properties 7 through 9 follow from the previous properties of limits. To prove the monomial property, use

$$\lim_{x\to a} cx^n \overset{\text{Const Mult}}{=} c[\lim_{x\to a} x^n] \overset{\text{Powers}}{=} c[\lim_{x\to a} x]^n \overset{\text{Thm 5.1}}{=} ca^n.$$

To prove the polynomial property, since $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a degree $n$ polynomial, then

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a}[c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0] \\
&\overset{\text{Sum}}{=} \lim_{x\to a} c_n x^n + \lim_{x\to a} c_{n-1}x^{n-1} + \cdots + \lim_{x\to a} c_1 x + \lim_{x\to a} c_0] \\
&\overset{\text{Monomial, Thm 5.2}}{=} c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

The rational function result is simpler, still. If $r(x) = \frac{p(x)}{q(x)}$ is a rational function, then $p(x)$ and $q(x)$ are polynomials. So for any point $a$ in the domain of $r(x)$ (i.e., $q(a) \neq 0$),

$$\lim_{x\to a} r(x) = \lim_{x\to a} \left(\frac{p(x)}{q(x)}\right) \overset{\text{Quotient}}{=} \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \overset{\text{Polynomial}}{=} \frac{p(a)}{q(a)} = r(a).$$

□

**EXAMPLE 4.1.13.** To see how these last results greatly simplify certain limit calculations, let's determine $\lim_{x\to 2} \frac{4x^2+2x}{3x+1}$.

**Solution.** Since we have a rational function and the denominator is not 0 at $x = 2$, we see that

$$\lim_{x\to 2} \frac{4x^2+2x}{3x+1} \overset{\text{Rational}}{=} \frac{4(2)^2+2(2)}{3(2)+1} = \frac{20}{7}.$$

That was easy!

*Several Cautions.* Most of the limits we will encounter this term will not be so easy to determine. While we will use the properties we've developed and others below, most limits will start off in the indeterminate form $\frac{0}{0}$. Typically we will need to carry out some sort of algebraic manipulation to get the limit in a form where the basic properties apply. For example, while

$$\lim_{x\to 5} \frac{x^2-25}{x-5}$$

is a rational function, property 8 above does not apply to the calculation of the limit since 5 (the number $x$ is approaching) is not in the domain of the function. Consequently, some algebraic manipulation (in this case factoring) is required.

$$\lim_{x\to 5}\frac{x^2-25}{x-5} = \lim_{x\to 5}\frac{(x-5)(x+5)}{x-5} = \lim_{x\to 5} x+5 \overset{\text{Poly}}{=} 10.$$

There are two additional things to notice. The first is *mathematical grammar*. We continue to use the limit symbol up until the actual numerical evaluation takes place. Writing something such as the following is simply wrong:

$$\xcancel{\lim_{x\to 5}\frac{x^2-25}{x-5} = \frac{(x-5)(x+5)}{x-5} = x+5 = 10.}$$

Among other things, the function $x+5$ is not the same as the constant 10.

An even worse calculation to write is

$$\xcancel{\lim_{x\to 5}\frac{x^2-25}{x-5} = \frac{0}{0} = 1}$$

or even

$$\xcancel{\lim_{x\to 5}\frac{x^2-25}{x-5} = \frac{0}{0} = \text{Undefined.}}$$

The expression $\frac{0}{0}$ is indeed not defined (and is certainly not equal to 1). However, the limit is indeterminate. Near (but not equal to) $x = 5$, the fraction is not yet $\frac{0}{0}$. You need to **do more work** to determine the limit. The work may involve factoring or other algebraic methods to simplify the expression so that we can more easily see what it is approaching.

Another thing to notice is that $\frac{x^2-25}{x-5}$ and $x+5$ are the *same* function as long as $x \neq 5$ where the first function is not defined but the second is. However, we are interested in a limit as $x \to 5$ so remember that this involves $x$ being close to, *but not equal to*, 5. Consequently $\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5}$ and $\lim\limits_{x\to 5} x+5$ are indeed the same!

**YOU TRY IT 4.4.** Determine each of the following limits, if they exist. Indicate the limit properties used at each step.

(*a*) $\lim\limits_{x\to 2}\sqrt{x^2-1}$ (*b*) $\lim\limits_{x\to -3}(2x^2-2x+3)^{4/3}$ (*c*) $\lim\limits_{x\to -3}(x^2-25)^{3/4}$

ANSWER TO YOU TRY IT ??:4.2-1.

(*a*) Notice that $\sqrt{x^2-1} = (x^2-1)^{1/2}$ is a fractional power function. In the language of Theorem 4.1.7, $\frac{n}{m} = \frac{1}{2}$ is reduced and $m = 2$ is even. Near $x = 2$, $f(x) = x^2-1$ is positive. So Theorem 4.1.7 applies and we may calculate the limit as

$$\lim_{x\to 2}\sqrt{x^2-1} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to 2} x^2-1\right)^{1/2} \overset{\text{Poly}}{=} (3)^{1/2} = \sqrt{3}.$$

(*b*) $(2x^2-2x+3)^{4/3}$ is a fractional power function with $\frac{n}{m} = \frac{4}{3}$ which is reduced and $m = 3$ is odd. So Theorem 4.1.7 applies and we may calculate the limit as

$$\lim_{x\to -3}(2x^2-2x+3)^{4/3} \overset{\text{Frac Pow}}{=} \left(\lim_{x\to -3} 2x^2-2x+3\right)^{4/3} \overset{\text{Sum, Prod}}{=} (18+6+3)^{4/3} = 27^{4/3} = 81.$$

(*c*) $(x^2-25)^{3/4}$ is a fractional power function. In the language of Theorem 4.1.7, $\frac{n}{m} = \frac{3}{4}$ is reduced and $m = 4$ is even. Near $x = -3$, $x^2-25$ is *negative*. Since $(x^2-25)^{3/4}$ is not even defined near $-3$, this limit does not exist.